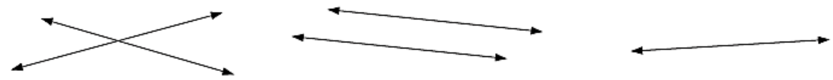
# COMMON QUESTIONS

## ● FINDING THE INTERSECTION OF TWO LINES

*REALTIONSHIP BETWEEN LINES*

**2 – D:**

| intersecting | parallel | coincident |
|---|---|---|



| one point of intersection<br>unique solution | lines do not meet<br>no solutions | the same line<br>infinitely many solutions |
|---|---|---|

**3 – D:**

- the lines are **coplanar** (they lie in the same plane). They could be:
  - intersecting
  - parallel
  - coincident
- the lines are **not coplanar** and are therefore **skew** (neither parallel nor intersecting)

**Q:** Line 1: $x = -1 + 2s, \quad y = 1 - 2s, \quad z = 1 + 4s$

Line 2: $x = 1 - t, \quad y = t, \quad z = 3 - 2t$

Line 3: $x = 1 + 2u, \quad y = -1 - u, \quad z = 4 + 3u$

*a*) Show that lines 2 and 3 intersect and find angle between them

*b*) Show that line 1 and 3 are skew.

**A:**

$a)\ 1 - t = 1 + 2u \Rightarrow t = -2u, \qquad t = -1 - u \Rightarrow -2u = -1 - u \Rightarrow u = 1\ \&\ t = -2$

$checking\ with\ z: 3 - 2t = 4 + 3u \Rightarrow 3 - 2(-2) = 4 + 3(1) \quad confirmed$

$\boxed{intersection\ (3, -2, 7)}$

$direction\ vectors\ for\ line\ 2\ and\ line\ 3\ are: \vec{b} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}\ and\ \vec{d} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$$\cos\theta = \frac{|\vec{b} \bullet \vec{d}|}{|\vec{b}||\vec{d}|} = \frac{|-2-1-6|}{\sqrt{1+1+4}\sqrt{4+1+9}} = \frac{9}{\sqrt{84}} \Rightarrow \boxed{\theta \approx 10.9^o}$$

$b)\ -1 + 2s = 1 + 2u \Rightarrow 2s - 2u = 2, \qquad 1 - 2s = -1 - u \Rightarrow -2s + u = -2, \qquad \Rightarrow u = 0\ \&\ s = 1$

$checking\ with\ z: \ 1 + 4s = 4 + 3u \Rightarrow 1 + 4(1) = 4 + 3(0) \Rightarrow 5 \neq 4$

$\Rightarrow no\ simultaneous\ solution\ to\ all\ 3\ equations$

$\therefore the\ line\ do\ not\ meet, and\ as\ they\ are\ not\ parallel\ (\vec{b} \neq k\vec{d}, k\epsilon R)\ \boxed{they\ must\ be\ skew.}$

**Q:** $Investigate\ relationship\ between\ \ lines: \overrightarrow{r_1} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}4\\5\\6\end{pmatrix}$ and $\overrightarrow{r_2} = \begin{pmatrix}1\\-2\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\0\\3\end{pmatrix}$

Are the lines

- the same?…….check by inspection (same direction and common point)
- parallel?………check by inspection (same direction)
- skew or do they have one point in common?
  solving $\mathbf{r_1} = \mathbf{r_2}$ will give 3 equations in λ and μ.

  Solve two of the equations for λ and μ

  if the values of λ and μ do not satisfy the third equation then the lines are skew, and they do not intersect.
  if these values do satisfy the three equations then substitute the value of λ or μ into the appropriate line and find the point of intersection.

## ● FINDING THE CARTESIAN EQUATION OF A PLANE GIVEN 3 POINTS

**Q:** Find the **Cartesian** equation of the plane passing through points $P_1(1,-1,4)$, $P_2(2,7,-1)$, and $P_3(5,0,-1)$.

**A:** *Method 1: Finding equation using normal vector*

*Two possible direction vectors are:* $\overrightarrow{P_1P_2} = \begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix}$ *and* $\overrightarrow{P_1P_3} = \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix}$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 8 & -5 \\ 4 & 1 & -5 \end{vmatrix} = \begin{pmatrix} -35 \\ -15 \\ -31 \end{pmatrix} \Rightarrow \vec{n} = \begin{pmatrix} 35 \\ 15 \\ 31 \end{pmatrix} \qquad \vec{n} \bullet \vec{r} = 35x + 15y + 31z \qquad \vec{a} \bullet \vec{n} = \begin{pmatrix} 35 \\ 15 \\ 31 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = 144$$

$$\boxed{35x + 15y + 31z = 144}$$

*one point on the plane is* $P_1 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$

**A:** *Method 2: Finding the vector equation and then eliminating the parameters*

*Two possible direction vectors* $\overrightarrow{P_1P_2} = \begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix}$ *and* $\overrightarrow{P_1P_3} = \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix}$ *and a point* $P_1 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \Rightarrow$ *vector form*

$$\vec{r} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix} \Rightarrow \begin{array}{ll} x = 1 + \lambda + 4\mu & (1) \\ y = -1 + 8\lambda + \mu & (2) \\ z = 4 - 5\lambda - 5\mu & (3) \end{array}$$

$$8(1) - (2) \Rightarrow 8x - y = 9 + 32\mu \ (4) \qquad \& \qquad 5(1) + (3) \Rightarrow 5x + z = 9 + 15\mu \ (5)$$

$$15(4) - 31(5) \Rightarrow \boxed{35x + 15y + 31z = 144}$$

## ● FINDING VECTOR FORM FROM CARTESIAN EQUATION OF A PLANE

It's fairly straightforward to convert a vector equation into a Cartesian equation, as you simply find the cross product of the two vectors appearing in the vector equation to find a normal to the plane and use that to find the Cartesian equation. But this process can't exactly be reversed to go the other way.
To convert Cartesian -> vector form, you need either two vectors or three points that lie on the plane!

**Q:** Convert the Cartesian equation of the plane $x - 2y + 2z = 5$ into

(a) a vector equation of the form $\vec{r} \bullet \hat{n} = D$, where $\hat{n}$ is a unit vector.
(b) vector form
(c) State the perpendicular distance of the plane from the origin.

**A:** (a) $\hat{n} = \frac{1}{\sqrt{1+4+4}} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix} = 5/3$

$5/3$ *is the perpendicular distance of the plane from the origin*

(b) choose three arbitrary random non-collinear points: $A(0, 1/2, 3)$ $B(0, 3/2, 4)$ $C(3, 0, 1)$

$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \overrightarrow{AC} = \begin{pmatrix} 3 \\ -1/2 \\ -2 \end{pmatrix} \Rightarrow \vec{r} = \begin{pmatrix} 0 \\ 1/2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1/2 \\ -2 \end{pmatrix}$$

(c) 5/3

## ● FINDING THE LINE OF INTERSECTION OF TWO PLANES

*Method 1: Solving two equations with three unknowns*

**Q:** Find intersection line: plane $\Pi_1$: $x+2y+3z=5$ and plane $\Pi_2$: $2x-2y-2z=2$

**A:** First checking if there is intersection:
The vector (1, 2, 3) is normal to the plane $\Pi_1$. The vector (2, -2, -2) is normal to the plane $\Pi_2$.
These vectors aren't parallel so the planes *do* meet!

When finding intersection be aware:
2 equations with 3 unknowns – meaning two coordinates will be expressed in the terms of the third one, and that will be true for **any** value of the third variable. I am choosing $x$ this time.

$$x+2y+3z=5$$
$$2x-2y-2z=2$$

$$2(EQ1)-EQ2 \Rightarrow \quad 6y+8z=8 \Rightarrow \quad \begin{matrix} y=-\frac{4}{3}z+\frac{4}{3} & into\ \ EQ1 \Rightarrow \\ z=-\frac{3}{4}y+1 & into\ EQ2 \Rightarrow \end{matrix}$$

$$\Rightarrow \quad x+2\left(-\frac{4}{3}z+\frac{4}{3}\right)+3z=5 \quad \Rightarrow \quad x+\frac{1}{3}z=\frac{7}{3} \quad \Rightarrow \quad z=-3x+7$$
$$\Rightarrow \quad 2x-2y-2\left(-\frac{3}{4}y+1\right)=2 \quad \Rightarrow \quad 2x-\frac{1}{2}y=4 \quad \Rightarrow \quad y=4x-8$$

These two equations are true for any value of x. It is a convention that we call such a variable "parameter" and label it with $t \in R$

$$x=t$$
$$y=4x-8=-8+4t$$
$$z=-3x+7=7-3t$$

In vector form is equation of the intersecting line is: $\vec{r}=\begin{pmatrix}0\\-8\\7\end{pmatrix}+t\begin{pmatrix}1\\4\\-3\end{pmatrix}$

(Equally yo
u can write t = y as a function of x and z, or t = z as a function of x and y).

*The same Method using Augmented Matrix Form*

$\Pi_1$: $x+2y+3z=5$ $\Pi_2$: $2x-2y-2z=2$
First, check by inspection, that the planes are not parallel (normal vectors are not parallel).
Find intersection:

$$\left(\begin{array}{ccc|c}0&0&0&0\\1&2&3&5\\2&-2&-2&2\end{array}\right)\sim\left(\begin{array}{ccc|c}0&0&0&0\\4&-1&0&8\\2&-2&-2&2\end{array}\right) \mapsto \quad x=t \qquad y=4x+8 \qquad z=-3x+7$$
$$\mapsto \quad t=x=\frac{y+8}{4}=\frac{z-7}{-3}$$

which is the equation of the common line, which in vector form is $\vec{r}=\begin{pmatrix}0\\-8\\7\end{pmatrix}+t\begin{pmatrix}1\\4\\-3\end{pmatrix}$
(Equally you can write t = y as a function of x and z, or t = z as a function of x and y).

*Method 2: Using cross product of two normal vectors as direction vector*

**Find the vector product of both normals to give the direction of the line. Then you need a point on the line .**

**Q:** Find the vector equation of the line in which the two planes 2x - 5y + 3z = 12 and 3x + 4y - 3z = 6 meet.

**A:** The vector (2, -5, 3) is normal to the plane 2x - 5y + 3z = 12. The vector (3, 4, -3) is normal to the plane 3x + 4y - 3z = 6.
These vectors aren't parallel so the planes *do* meet!

The cross product of these two normal vectors gives a vector which is perpendicular to both of them and which is therefore parallel to the line of intersection of the two planes. So this cross product will give a direction vector for the line of intersection.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -5 & 3 \\ 3 & 4 & -3 \end{vmatrix} = \begin{pmatrix} 3 \\ 15 \\ 23 \end{pmatrix}$$

Any point which lies on both planes will do as a point A on the line. I can see that both planes will have points for which x = 0.

These points in 2x - 5y + 3z = 12 will have -5y + 3z = 12.
These points in 3x + 4y - 3z = 6 will have 4y - 3z = 6.

Solving these two equations simultaneously gives y = -18 and z = -26 so the point with position vector (0, -18, -26) lies on the line of intersection.
Therefore the equation of the line of intersection is

$$\vec{r} = \begin{pmatrix} 0 \\ -18 \\ -26 \end{pmatrix} + t\begin{pmatrix} 3 \\ 15 \\ 23 \end{pmatrix}$$

## ● FINDING THE INTERSECTION OF A PLANE AND A LINE

There are several possibilities:

- the line could lie within the plane;
- the line could intersect the plane at a single point;
- the line could be parallel to the plane.

**Q:** Do line L: $\vec{r} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + \mu\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ and plane Π: $x + 2y + 3z = 5$ intersect?

**A:** First check dot product of direction vector of the line and normal vector of the plane: $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 32$

therefore the line and the plane are not parallel and the line will intersect the plane in one point.

Substitute the line equation into the plane equation to obtain the value of the line parameter, μ.

$$(1 + 4\mu) + 2(-2 + 5\mu) + 3(-1 + 6\mu) = 5 \;\Rightarrow\; 1 + 4\mu - 4 + 10\mu - 3 + 18\mu = 5 \quad \Rightarrow \quad \mu = \frac{11}{32}$$

Substitute μ into the equation of the line to obtain the co-ordinates of the point of intersection. $\frac{1}{32}\begin{pmatrix} 76 \\ -9 \\ 34 \end{pmatrix}$

In general: Solve for μ and substitute into the equation of the line to get the point of intersection. If this equation gives you something like 0 = 5, then the line will be parallel and not in the plane, and if the equation gives you something like 5 = 5 then the line is contained in the plane.

**Q:** Investigate which of the three situations above applies with the line $\frac{x-5}{-1}=\frac{y-9}{6}=\frac{z-1}{2}$ and the plane $2x - y + 4z = 5$.

**A:** To investigate this situation, it is simplest if we have the equation of the line in vector form and the equation of the plane in Cartesian form (which it is already).

The vector equation of the line is: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 6 \\ 2 \end{pmatrix}$

i.e. $x = 5 - t$
$y = 9 + 6t$
$z = 1 + 2t$

Substituting these into the equation of the plane gives:

$2x - y + 4z = 5$
$2(5 - t) - (9 + 6t) + 4(1 + 2t) = 5$
$10 - 2t - 9 - 6t + 4 + 8t = 5$
$5 = 5$

This equation is not true for some specific t. Rather it is true for all values of $t$. Therefore the line must lie within the plane.

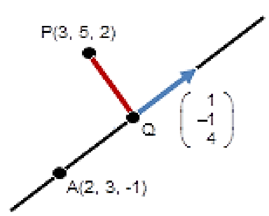
Note 1: If when we attempt to solve the equations, we get an equation which is true for no values of $t$ (such as 4 = -2), we know that the plane and the line do not intersect, i.e. are parallel.

Note 2: If when we attempt to solve the equations, we get an equation which is true for a single value of $t$, we know that the plane and the line intersect at a single point. We can find the point of intersection by substituting the value of $t$ into the vector equation of the line.

## • FINDING THE PERPENDICULAR DISTANCE FROM A POINT TO A LINE

**Q:** Find the perpendicular distance from the point P(3, 5, 2) to the line with equation

$$\vec{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$




**A:** Let Q be the closest point on the line to P. It lies on the line:

$$\begin{pmatrix} x_Q \\ y_Q \\ z_Q \end{pmatrix} = \begin{pmatrix} 2+t \\ 3-t \\ -1+4t \end{pmatrix}$$

Then $\overrightarrow{PQ}$ is:

$$\overrightarrow{PQ} = \begin{pmatrix} -1+t \\ -2-t \\ -3+4t \end{pmatrix}$$

Then PQ is perpendicular to the direction of the line.

$$\overrightarrow{PQ} \bullet \vec{b} = 0 \quad \Rightarrow \quad \begin{pmatrix} -1+t \\ -2-t \\ -3+4t \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = 0 \quad \Rightarrow \quad -1+t+2+t-12+16t = 0 \quad \Rightarrow \quad t = \frac{11}{18}$$

$$\overrightarrow{PQ} = -\frac{1}{18}\begin{pmatrix} 7 \\ 47 \\ 10 \end{pmatrix} \quad \Rightarrow \quad |\overrightarrow{PQ}| = \frac{1}{18}\sqrt{49+2209+100}$$

$$|\overrightarrow{PQ}| = 2.70$$

## ● FINDING THE PERPENDICULAR DISTANCE FROM THE ORIGIN TO A PLANE

**Q:** Find the distance of the plane $\vec{r}\bullet\begin{pmatrix}3\\2\\-4\end{pmatrix} = 8$ $\left[\,3x + 2y - 4z = 8\,\right]$ from the origin,

and the unit vector perpendicular to the plane.

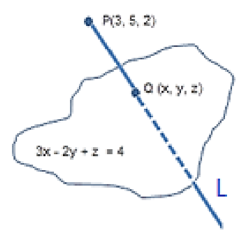
**A:** $\vec{n} = \begin{pmatrix}3\\2\\-4\end{pmatrix} \Rightarrow \left|\begin{pmatrix}3\\2\\-4\end{pmatrix}\right| = \sqrt{29}$

$$\boxed{D = \frac{8}{\sqrt{29}}} \qquad \boxed{\hat{n} = \frac{1}{\sqrt{29}}\begin{pmatrix}3\\2\\-4\end{pmatrix}}$$

## ● FINDING THE PERPENDICULAR DISTANCE FROM A POINT TO A PLANE

**Q:** Find the perpendicular distance from the point P(3, 5, 2) to the plane with equation $3x - 2y + z = 4$

**A:** Let Q be the foot of the perpendicular from point P to the plane.
Then PQ will be perpendicular to the plane.




$$\vec{n} = \begin{pmatrix}3\\-2\\1\end{pmatrix} \textit{ is perpendicular to the plane}$$

$$\Rightarrow \overrightarrow{PQ} = t\begin{pmatrix}3\\-2\\1\end{pmatrix}$$

*L (PQ) perpendicular to plane and passing through P is*

$$\vec{r} = \begin{pmatrix}3\\5\\2\end{pmatrix} + t\begin{pmatrix}3\\-2\\1\end{pmatrix}$$

Q is intersection of the line L and plane

Substituting $x = 3 + 3t$, $y = 5 - 2t$, $z = 2 + t$ into the equation of the plane, gives:

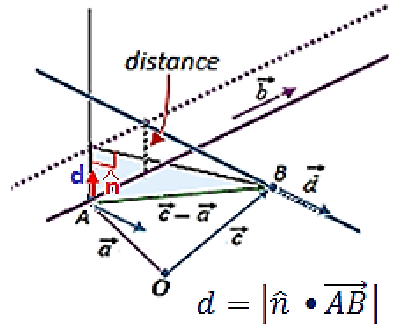
$$3(3 + 3t) - 2(5 - 2t) + (2 + t) = 4 \Rightarrow 9 + 9t - 10 + 4t + 2 + t = 4 \Rightarrow t = \frac{3}{14}$$

$$\Rightarrow \overrightarrow{PQ} = \frac{3}{14}\begin{pmatrix}3\\-2\\1\end{pmatrix} \Rightarrow |\overrightarrow{PQ}| = \frac{3}{14}\sqrt{14} \Rightarrow \boxed{|\overrightarrow{PQ}| = 0.802}$$

## ● FINDING THE SHORTEST DISTANCE BETWEEN TWO SKEW LINES

*The shortest distance between two skew lines* $\vec{r} = \vec{a} + t\,\vec{b}$ *and* $\vec{r} = \vec{c} + s\,\vec{d}$ *is:*

$$d = |\hat{n} \bullet (\vec{c} - \vec{a})| \quad where \quad \hat{n} = \frac{\vec{b} \times \vec{d}}{|\vec{b} \times \vec{d}|}$$




**Q**: Find the shortest distance between the lines

$$\vec{r} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad and \quad \vec{r} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} + s \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

**A:**

$$\vec{n} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} = \begin{pmatrix} -1 \\ 7 \\ 4 \end{pmatrix} \quad \Rightarrow \quad \hat{n} = \frac{1}{\sqrt{66}} \begin{pmatrix} -1 \\ 7 \\ 4 \end{pmatrix}$$

$$d = |\hat{n} \bullet (\vec{c} - \vec{a})| = \left| \frac{1}{\sqrt{66}} \begin{pmatrix} -1 \\ 7 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -4 \\ -6 \end{pmatrix} \right| = \left| \frac{-53}{\sqrt{66}} \right| \quad \Rightarrow \boxed{d = 6.52}$$

## ● FINDING THE ANGLE BETWEEN TWO PLANES

*The angle between two planes is the same as the angle between the two normal vectors. (acute angle)*

Q: Find the acute angle between the planes with equations: $\vec{r} \bullet \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = 2 \quad and \quad \vec{r} \bullet \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = -1$

A: The equations of the planes are $3x + 0y - z = 2$ and $x + 2y + 5z = -1$.

The normal vectors are $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$

The scalar product is: $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = -2$

The magnitudes are: $\left| \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \right| = \sqrt{10} \quad and \quad \left| \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \right| = \sqrt{30}$

$$\theta = arc\,cos \frac{|\vec{n} \bullet \vec{m}|}{|\vec{n}||\vec{m}|} = arc\,cos \frac{2}{\sqrt{300}} \quad \Rightarrow \quad \boxed{\theta = 83.4^0}$$

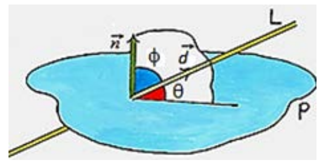
## ● FINDING THE ANGLE BETWEEN A PLANE AND A LINE

To find the angle between a plane and a line, there are 2 steps:
Step 1: Find the angle between the normal vector and the direction vector of the line;
Step 2: Subtract the angle from step 1 from 90° in order to get the angle required.

**Q:** Find the angle between the line $\frac{x+2}{2}=\frac{y}{1}=\frac{z-3}{-1}$ and the plane $\vec{r}\bullet\begin{pmatrix}4\\3\\-2\end{pmatrix}=5 \quad [4x+3y-2z=5\,]$ .

**A:**

First we find the angle between the vectors $\begin{pmatrix}2\\1\\-1\end{pmatrix}$ *and* $\begin{pmatrix}4\\3\\-2\end{pmatrix}$




$$\cos\phi=\frac{|\vec{n}\bullet\vec{d}|}{|\vec{n}||\vec{d}|}=\frac{13}{\sqrt{6}\sqrt{29}}=9.76^0$$

$$\boxed{\theta=80.2^0}$$