AP® Calculus AB
AP® Calculus BC

Free-Response Questions and Solutions 1989 – 1997

1989 BC1

Let f be a function such that $f''(x) = 6x + 8$.

(a) Find $f(x)$ if the graph of f is tangent to the line $3x - y = 2$ at the point $(0, -2)$.

(b) Find the average value of $f(x)$ on the closed interval $[-1, 1]$.

1989 BC1
Solution

(a) $f'(x) = 3x^2 + 8x + C$

$f'(0) = 3$

$C = 3$

$f(x) = x^3 + 4x^2 + 3x + d$

$d = -2$

$f(x) = x^3 + 4x^2 + 3x - 2$

(b) $\dfrac{1}{1-(-1)}\int_{-1}^{1}\left(x^3 + 4x^2 + 3x - 2\right)dx$

$= \dfrac{1}{2}\left[\dfrac{1}{4}x^4 + \dfrac{4}{3}x^3 + \dfrac{3}{2}x^2 - 2x\right]\Bigg|_{-1}^{1}$

$= \dfrac{1}{2}\left[\left(\dfrac{1}{4} + \dfrac{4}{3} + \dfrac{3}{2} - 2\right) - \left(\dfrac{1}{4} - \dfrac{4}{3} + \dfrac{3}{2} + 2\right)\right]$

$= -\dfrac{2}{3}$

1989 BC2

Let R be the region enclosed by the graph of $y = \frac{x^2}{x^2+1}$, the line $x = 1$, and the x-axis.

(a) Find the area of R .

(b) Find the volume of the solid generated when R is rotated about the <u>y-axis</u>.

1989 BC2
Solution

(a) $$\text{Area} = \int_0^1 \frac{x^2}{x^2+1}\,dx$$
$$= \int_0^1 1 - \frac{1}{x^2+1}\,dx$$
$$= x - \arctan x \Big|_0^1$$
$$= 1 - \frac{\pi}{4}$$

(b) $$\text{Volume} = 2\pi \int_0^1 x\left(\frac{x^2}{x^2+1}\right)dx$$
$$= 2\pi \int_0^1 x - \frac{x}{x^2+1}\,dx$$
$$= 2\pi\left(\frac{x^2}{2} - \frac{1}{2}\ln\left|x^2+1\right|\right)\Bigg|_0^1$$
$$= \pi\left(1 - \ln 2\right)$$

<u>or</u>

$$\text{Volume} = \pi \int_0^{1/2} \left(1 - \frac{y}{1-y}\right)dy$$
$$= \pi\left(2y + \ln\left|y-1\right|\right)\Big|_0^{1/2}$$
$$= \pi\left(1 - \ln 2\right)$$

1989 BC3

Consider the function f defined by $f(x) = e^x \cos x$ with domain $[0, 2\pi]$.

(a) Find the absolute maximum and minimum values of $f(x)$.

(b) Find the intervals on which f is increasing.

(c) Find the x-coordinate of each point of inflection of the graph of f.

1989 BC3
Solution

(a) $f'(x) = -e^x \sin x + e^x \cos x$

$= e^x [\cos x - \sin x]$

$f'(x) = 0$ when $\sin x = \cos x$, $x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

x	$f(x)$
0	1
$\dfrac{\pi}{4}$	$\dfrac{\sqrt{2}}{2} e^{\pi/4}$
$\dfrac{5\pi}{4}$	$-\dfrac{\sqrt{2}}{2} e^{5\pi/4}$
2π	$e^{2\pi}$

Max: $e^{2\pi}$; Min: $-\dfrac{\sqrt{2}}{2} e^{5\pi/4}$

(b) $f'(x)$: $+$ on $\left(0, \dfrac{\pi}{4}\right)$, $-$ on $\left(\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right)$, $+$ on $\left(\dfrac{5\pi}{4}, 2\pi\right)$

Increasing on $\left[0, \dfrac{\pi}{4}\right]$, $\left[\dfrac{5\pi}{4}, 2\pi\right]$

(c) $f''(x) = e^x [-\sin x - \cos x] + e^x [\cos x - \sin x]$

$= -2e^x \sin x$

$f''(x) = 0$ when $x = 0, \pi, 2\pi$

Point of inflection at $x = \pi$

1989 BC4

Consider the curve given by the parametric equations

$$x = 2t^3 - 3t^2 \quad \text{and} \quad y = t^3 - 12t$$

(a) In terms of t, find $\dfrac{dy}{dx}$.

(b) Write an equation for the line tangent to the curve at the point where $t = -1$.

(c) Find the x- and y-coordinates for each critical point on the curve and identify each point as having a vertical or horizontal tangent.

1989 BC4
Solution

(a) $\dfrac{dy}{dt} = 3t^2 - 12$

$\dfrac{dx}{dt} = 6t^2 - 6t$

$\dfrac{dy}{dx} = \dfrac{3t^2 - 12}{6t^2 - 6t} = \dfrac{t^2 - 4}{2t^2 - 2t} = \dfrac{(t+2)(t-2)}{2t(t-1)}$

(b) $x = -5,\ \ y = 11$

$\dfrac{dy}{dx} = -\dfrac{3}{4}$

$y - 11 = -\dfrac{3}{4}(x+5)$

or

$y = -\dfrac{3}{4}x + \dfrac{29}{4}$

$4y + 3x = 29$

(c)

t	(x, y)	type
-2	$(-28, 16)$	horizontal
0	$(0, 0)$	vertical
1	$(-1, -11)$	vertical
2	$(4, -16)$	horizontal

1989 BC5

At any time $t \geq 0$, the velocity of a particle traveling along the x-axis is given by the differential equation $\frac{dx}{dt} - 10x = 60e^{4t}$.

(a) Find the general solution $x(t)$ for the position of the particle.

(b) If the position of the particle at time $t = 0$ is $x = -8$, find the particular solution $x(t)$ for the position of the particle.

(c) Use the particular solution from part (b) to find the time at which the particle is at rest.

1989 BC5
Solution

(a) Integrating Factor: $e^{-\int 10\,dt} = e^{-10t}$

$$\frac{d}{dt}\left(xe^{-10t}\right) = 60e^{4t}e^{-10t}$$
$$xe^{-10t} = -10e^{-6t} + C$$
$$x(t) = -10e^{4t} + Ce^{10t}$$

<u>or</u>

$$x_h(t) = Ce^{10t}$$
$$x_p = Ae^{4t}$$
$$4Ae^{4t} - 10Ae^{4t} = 60e^{4t}$$
$$A = -10$$
$$x(t) = Ce^{10t} - 10e^{4t}$$

(b) $-8 = C - 10;\ C = 2$

$$x(t) = 2e^{10t} - 10e^{4t}$$

(c) $\dfrac{dx}{dt} = 20e^{10t} - 40e^{4t}$

$$20e^{10t} - 40e^{4t} = 0$$
$$t = \frac{1}{6}\ln 2$$

<u>or</u>

$$\frac{dx}{dt} - 10\left(-10e^{4t} + 2e^{10t}\right) = 60e^{4t}$$
$$0 + 100e^{4t} - 20e^{10t} = 60e^{4t}$$
$$t = \frac{1}{6}\ln 2$$

1989 BC6

Let f be a function that is everywhere differentiable and that has the following properties.

(i) $f(x+h) = \dfrac{f(x)+f(h)}{f(-x)+f(-h)}$ for all real numbers h and x.

(ii) $f(x) > 0$ for all real numbers x.

(iii) $f'(0) = -1$.

(a) Find the value of $f(0)$.

(b) Show that $f(-x) = \dfrac{1}{f(x)}$ for all real numbers x.

(c) Using part (b), show that $f(x+h) = f(x)f(h)$ for all real numbers h and x.

(d) Use the definition of the derivative to find $f'(x)$ in terms of $f(x)$.

1989 BC6
Solution

(a) Let $x = h = 0$

$$f(0) = f(0+0) = \frac{f(0)+f(0)}{f(0)+f(0)} = 1$$

(b) Let $h = 0$

$$f(x+0) = f(x) = \frac{f(x)+f(0)}{f(-x)+f(-0)}$$

Use $f(0) = 1$ and solve for $f(x) = \dfrac{1}{f(-x)}$

<u>or</u>

Note that $f(-x+0) = \dfrac{f(-x)+f(0)}{f(x)+f(0)}$ is the reciprocal of $f(x)$.

(c) $f(x+h) = \dfrac{f(x)+f(h)}{\dfrac{1}{f(x)}+\dfrac{1}{f(h)}}$

$$= \frac{f(x)+f(h)}{f(h)+f(x)} f(x) f(h)$$

$$= f(x) f(h)$$

(d) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x)f(h)-f(x)}{h}$$

$$= f(x) \lim_{h\to 0} \frac{f(h)-1}{h}$$

$$= f(x) f'(0) = -f(x)$$

1990 BC1

A particle starts at time $t = 0$ and moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = (t-1)^3(2t-3)$.

(a) Find the velocity of the particle at any time $t \geq 0$.

(b) For what values of t is the velocity of the particle less than zero?

(c) Find the value of t when the particle is moving and the acceleration is zero.

1990 BC1
Solution

(a) $v(t) = x'(t)$

$$= 3(t-1)^2(2t-3) + 2(t-1)^3$$

$$= (t-1)^2(8t-11)$$

(b) $v(t) < 0$ when $(t-1)^2(8t-11) < 0$

Therefore $8t - 11 < 0$ and $t \neq 1$

or $t < \dfrac{11}{8}$ and $t \neq 1$

Since $t \geq 0,$ answer is $0 \leq t < \dfrac{11}{8},$ except $t = 1$

(c) $a(t) = v'(t)$

$$= 2(t-1)(8t-11) + 8(t-1)^2$$

$$= 6(t-1)(4t-5)$$

$a(t) = 0$ when $t = 1, \quad t = \dfrac{5}{4}$

but particle not moving at $t = 1$ so $t = \dfrac{5}{4}$

1990 BC2

Let R be the region in the xy-plane between the graphs of $\boldsymbol{y = e^{x}}$ and $\boldsymbol{y = e^{-x}}$ from $x = 0$ to $x = 2$.

(a) Find the volume of the solid generated when R is revolved about the x-axis.

(b) Find the volume of the solid generated when R is revolved about the y-axis.

1990 BC2
Solution

(a) $V = \pi \int_0^2 \left(e^{2x} - e^{-2x}\right) dx$

$$= \pi \left[\frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x} \right]_0^2$$

$$= \pi \left[\frac{1}{2} e^{4} + \frac{1}{2} e^{-4} - \left(\frac{1}{2} + \frac{1}{2} \right) \right]$$

$$= \frac{\pi}{2} \left[e^4 + e^{-4} - 2 \right]$$

(b) $V = 2\pi \int_0^2 x \left[e^x - e^{-x} \right] dx$

$$= 2\pi \left[x\left(e^x + e^{-x}\right) - \int e^x + e^{-x}\, dx \right]_0^2$$

$$= 2\pi \left[x\left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right) \right]_0^2$$

$$= 2\pi \left[2\left(e^2 + e^{-2}\right) - \left(e^2 - e^{-2}\right) - \left[0 - (1 - 1)\right] \right]$$

$$= 2\pi \left[e^2 + 3e^{-2} \right]$$

1990 BC3

Let $f(x) = 12 - x^2$ for $x \geq 0$ and $f(x) \geq 0$.

(a) The line tangent to the graph of f at the point $(k, f(k))$ intercepts the x-axis at $x = 4$. What is the value of k ?

(b) An isosceles triangle whose base is the interval from $(0,0)$ to $(c,0)$ has its vertex on the graph of f. For what value of c does the triangle have maximum area? Justify your answer.

1990 BC3
Solution

(a) $f(x) = 12 - x^2$; $f'(x) = -2x$

slope of tangent line at

$(k, f(k)) = -2k$

line through $(4,0)$ & $(k, f(k))$ has slope

$$\frac{f(k) - 0}{k-4} = \frac{12-k^2}{k-4}$$

so $-2k = \dfrac{12-k^2}{k-4} \Rightarrow k^2 - 8k + 12 = 0$

$k = 2$ or $k = 6$ but $f(6) = -24$

so 6 is not in the domain.

$k = 2$

(b) $A = \dfrac{1}{2} c \cdot f\left(\dfrac{c}{2}\right) = \dfrac{1}{2} c\left(12 - \dfrac{c^2}{4}\right)$

$= 6c - \dfrac{c^3}{8}$ on $\left[0, 4\sqrt{3}\right]$

$\dfrac{dA}{dc} = 6 - \dfrac{3c^2}{8}$; $6 - \dfrac{3c^2}{8} = 0$ when c = 4.

Candidate test

c	A	
0	0	
4	16	← Max
$4\sqrt{3}$	0	

First derivative

A'		+		−	
	0		4		$4\sqrt{3}$

second derivative

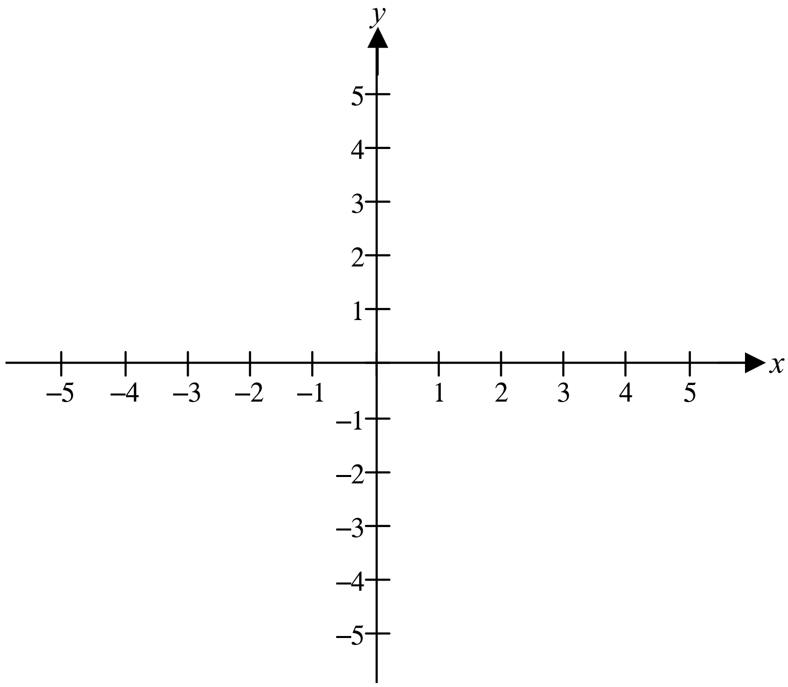
$\left.\dfrac{d^2A}{dc^2}\right|_{c=4} = -3 < 0$ so $c = 4$ gives a relative max.

$c = 4$ is the only critical value in the domain interval, therefore maximum

1990 BC4

Let R be the region inside the graph of the polar curve $r = 2$ and outside the graph of the polar curve $r = 2(1 - \sin\theta)$.

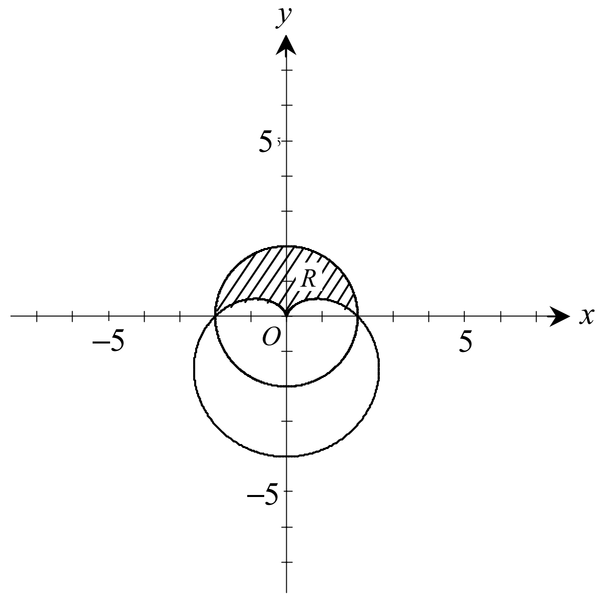
(a) Sketch the two polar curves in the xy-plane provided below and shade the region R .



(b) Find the area of R .

1990 BC4
Solution

(a)



(b) $A = \frac{1}{2}\int_0^{\pi}\left[2^2 - \left(2\left(1-\sin\theta\right)\right)^2\right]d\theta$

$= 2\int_0^{\pi}\left(2\sin\theta - \sin^2\theta\right)d\theta$

$= 4\int_0^{\pi}\sin\theta\, d\theta - \int_0^{\pi}\left(1-\cos 2\theta\right)d\theta$

$= -4\cos\theta\Big|_0^{\pi} - \left[\theta - \frac{1}{2}\sin 2\theta\right]\Big|_0^{\pi}$

$= \left[-4(-1)+4(1)\right] - \left[\pi - 0\right]$

$= 8-\pi$

1990 BC5

Let f be the function defined by $f(x) = \frac{1}{x-1}$.

(a) Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x = 2$.

(b) Use the result from part (a) to find the first four terms and the general term of the series expansion about $x = 2$ for $\ln|x-1|$.

(c) Use the series in part (b) to compute a number that differs from $\ln\frac{3}{2}$ by less than 0.05. Justify your answer.

1990 BC5
Solution

(a) Taylor approach

$f(2)=1$

$f'(2)=-(2-1)^{-2}=-1$

$f''(2)=2(2-1)^{-3}=2;\quad \dfrac{f''(2)}{2!}=1$

$f'''(2)=-6(2-1)^{-4}=-6;\ \dfrac{f'''(2)}{3!}=-1$

Therefore $\dfrac{1}{x-1}=1-(x-2)+(x-2)^2-(x-2)^3+\cdots+(-1)^n(x-2)^n+\cdots$

Geometric Approach

$$\frac{1}{x-1}=\frac{1}{1+(x-2)}$$

$$=1-u+u^2-u^3+\cdots+(-1)^n u^n+\cdots$$

where $u=x-2$

(b) Antidifferentiates series in (a):

$$\ln|x-1|=C+x-\frac{1}{2}(x-2)^2+\frac{1}{3}(x-2)^3-\frac{1}{4}(x-2)^4+\cdots+\frac{(-1)^n(x-2)^{n+1}}{n+1}+\cdots$$

$$0=\ln|2-1|\Rightarrow C=-2$$

Note: If $C\neq 0$, “first 4 terms” need not include $-\frac{1}{4}(x-2)^4$

(c) $$\ln\frac{3}{2}=\ln\left|\frac{5}{2}-1\right|=\frac{1}{2}-\frac{1}{2}\left(\frac{1}{2}\right)^2+\frac{1}{3}\left(\frac{1}{2}\right)^3-\cdots$$

$$=\frac{1}{2}-\frac{1}{8}+\frac{1}{24}-\cdots$$

since $\frac{1}{24}<\frac{1}{20}$, $\frac{1}{2}-\frac{1}{8}=0.375$ is sufficient.

Justification: Since series is alternating, with terms convergent to 0 and decreasing in absolute value, the truncation error is less than the first omitted term.

Alternate Justification: $$|R_n|=\left|\frac{1}{(C-1)^{n+1}}\frac{1}{n+1}\left(\frac{1}{2}\right)^{n+1}\right|,\text{ where } 2<C<\frac{5}{2}$$

$$<\frac{1}{n+1}\frac{1}{2^{n+1}}$$

$$<\frac{1}{20}\text{ when } n\geq 2$$

1990 BC6

Let f and g be continuous functions with the following properties.

(i) $g(x) = A - f(x)$ where A is a constant

(ii) $\int_1^2 f(x)\,dx = \int_2^3 g(x)\,dx$

(iii) $\int_2^3 f(x)\,dx = -3A$

(a) Find $\int_1^3 f(x)\,dx$ in terms of A .

(b) Find the average value of $g(x)$ in terms of A , over the interval $[1,3]$.

(c) Find the value of k if $\int_0^1 f(x+1)\,dx = k\,A$.

1990 BC6
Solution

(a) $\int_1^3 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$

$= \int_2^3 g(x)\,dx + \int_2^3 f(x)\,dx$

$= \int_2^3 \big(A - f(x)\big)\,dx + \int_2^3 f(x)\,dx$

$= A - \int_2^3 f(x)\,dx + \int_2^3 f(x)\,dx$

$= A$

(b) Average value $= \frac{1}{2}\int_1^3 g(x)\,dx = \frac{1}{2}\int_1^3 \big(A - f(x)\big)\,dx$

$= \frac{1}{2}\left[2A - \int_1^3 f(x)\,dx\right]$

$= \frac{1}{2}[2A - A] = \frac{1}{2}A$

(c) $kA = \int_0^1 f(x+1)\,dx = \int_1^2 f(x)\,dx$

$= \int_2^3 g(x)\,dx$

$= A + 3A = 4A$

Therefore $k = 4$

1990 BC5

Let f be the function defined by $f(x) = \dfrac{1}{x-1}$.

(a) Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x = 2$.

(b) Use the result from part (a) to find the first four terms and the general term of the series expansion about $x = 2$ for $\ln|x-1|$.

(c) Use the series in part (b) to compute a number that differs from $\ln\dfrac{3}{2}$ by less than 0.05. Justify your answer.

1990 BC5
Solution

(a) Taylor approach

$f(2)=1$

$f'(2)=-(2-1)^{-2}=-1$

$f''(2)=2(2-1)^{-3}=2; \quad \frac{f''(2)}{2!}=1$

$f'''(2)=-6(2-1)^{-4}=-6; \frac{f'''(2)}{3!}=-1$

Geometric Approach

$$\frac{1}{x-1}=\frac{1}{1+(x-2)}$$
$$=1-u+u^2-u^3+\cdots+(-1)^n u^n+\cdots$$
where $u=x-2$

Therefore $\frac{1}{x-1}=1-(x-2)+(x-2)^2-(x-2)^3+\cdots+(-1)^n(x-2)^n+\cdots$

(b) Antidifferentiates series in (a):

$$\ln|x-1|=C+x-\frac{1}{2}(x-2)^2+\frac{1}{3}(x-2)^3-\frac{1}{4}(x-2)^4+\cdots+\frac{(-1)^n(x-2)^{n+1}}{n+1}+\cdots$$
$$0=\ln|2-1|\Rightarrow C=-2$$

Note: If $C\neq 0$, "first 4 terms" need not include $-\frac{1}{4}(x-2)^4$

(c) $$\ln\frac{3}{2}=\ln\left|\frac{5}{2}-1\right|=\frac{1}{2}-\frac{1}{2}\left(\frac{1}{2}\right)^2+\frac{1}{3}\left(\frac{1}{2}\right)^3-\cdots$$
$$=\frac{1}{2}-\frac{1}{8}+\frac{1}{24}-\cdots$$

since $\frac{1}{24}<\frac{1}{20}$, $\frac{1}{2}-\frac{1}{8}=0.375$ is sufficient.

Justification: Since series is alternating, with terms convergent to 0 and decreasing in absolute value, the truncation error is less than the first omitted term.

Alternate Justification: $$|R_n|=\left|\frac{1}{(C-1)^{n+1}}\frac{1}{n+1}\left(\frac{1}{2}\right)^{n+1}\right|, \text{ where } 2<C<\frac{5}{2}$$
$$<\frac{1}{n+1}\frac{1}{2^{n+1}}$$
$$<\frac{1}{20} \text{ when } n\geq 2$$

1990 BC6

Let f and g be continuous functions with the following properties.

(i) $g(x) = A - f(x)$ where A is a constant

(ii) $\int_1^2 f(x)\,dx = \int_2^3 g(x)\,dx$

(iii) $\int_2^3 f(x)\,dx = -3A$

(a) Find $\int_1^3 f(x)\,dx$ in terms of A .

(b) Find the average value of $g(x)$ in terms of A , over the interval $[1,3]$.

(c) Find the value of k if $\int_0^1 f(x+1)\,dx = k\,A$.

1990 BC6
Solution

(a) $$\int_1^3 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$
$$= \int_2^3 g(x)\,dx + \int_2^3 f(x)\,dx$$
$$= \int_2^3 \left(A - f(x)\right)dx + \int_2^3 f(x)\,dx$$
$$= A - \int_2^3 f(x)\,dx + \int_2^3 f(x)\,dx$$
$$= A$$

(b) Average value $= \dfrac{1}{2}\displaystyle\int_1^3 g(x)\,dx = \dfrac{1}{2}\int_1^3 \left(A - f(x)\right)dx$
$$= \frac{1}{2}\left[2A - \int_1^3 f(x)\,dx\right]$$
$$= \frac{1}{2}[2A - A] = \frac{1}{2}A$$

(c) $$kA = \int_0^1 f(x+1)\,dx = \int_1^2 f(x)\,dx$$
$$= \int_2^3 g(x)\,dx$$
$$= A + 3A = 4A$$

Therefore $k = 4$

1991 BC1

A particle moves on the x-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = 12t^2 - 36t + 15$. At $t = 1$, the particle is at the origin.

(a) Find the position $x(t)$ of the particle at any time $t \geq 0$.

(b) Find all values of t for which the particle is at rest.

(c) Find the maximum velocity of the particle for $0 \leq t \leq 2$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

1991 BC1
Solution

(a) $x(t) = 4t^3 - 18t^2 + 15t + C$

$0 = x(1) = 4 - 18 + 15 + C$

Therefore $C = -1$

$x(t) = 4t^3 - 18t^2 + 15t - 1$

(b) $0 = v(t) = 12t^2 - 36t + 15$

$3(2t-1)(2t-5) = 0$

$t = \frac{1}{2}, \frac{5}{2}$

(c) $\frac{dv}{dt} = 24t - 36$

$\frac{dv}{dt} = 0$ when $t = \frac{3}{2}$

$v(0) = 15$

$v\left(\frac{3}{2}\right) = -12$

$v(2) = -9$

Maximum velocity is 15

(d) Total distance $= \int_0^{1/2} v(t)\,dt - \int_{1/2}^{2} v(t)\,dt$

$$= \left(x\left(\frac{1}{2}\right) - x(0)\right) - \left(x(2) - x\left(\frac{1}{2}\right)\right)$$

$$= \frac{5}{2} - (-1) - \left(-11 - \frac{5}{2}\right) = 17$$

1991 BC2

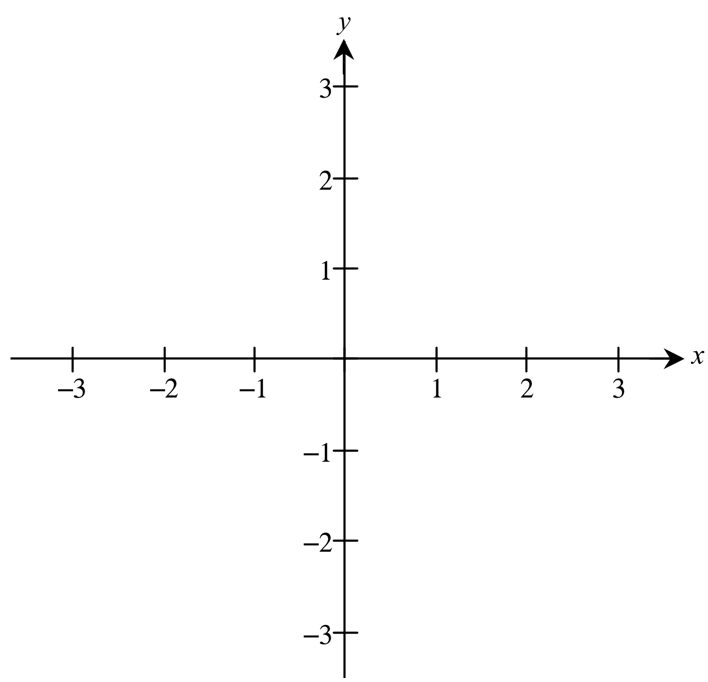
Let f be the function defined by $f(x) = xe^{1-x}$ for all real numbers x .

(a) Find each interval on which f is increasing.

(b) Find the range of f .

(c) Find the x-coordinate of each point of inflection of the graph of f .

(d) Using the results found in parts (a), (b), and (c), sketch the graph of f in the xy-plane provided below. (Indicate all intercepts.)



1991 BC2
Solution

(a) $f'(x) = xe^{1-x}(-1) + e^{1-x} = (1-x)e^{1-x}$

f increases on $(-\infty, 1]$

(b) $f(1) = 1;\ \lim_{x \to -\infty} f(x) = -\infty$

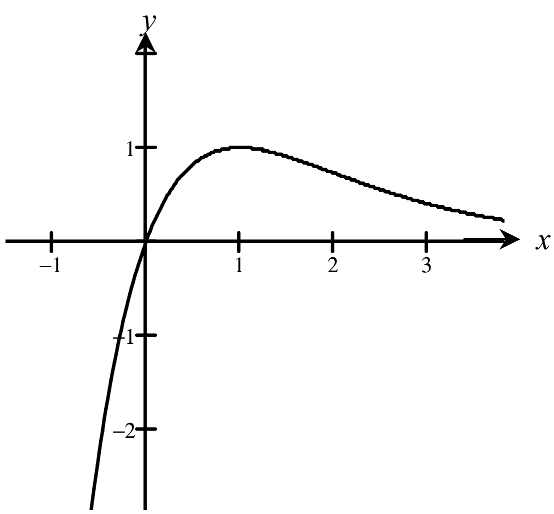
Range: $(-\infty, 1]$

(c) $f''(x) = e^{1-x}(-1) + (1-x)e^{1-x}(-1)$

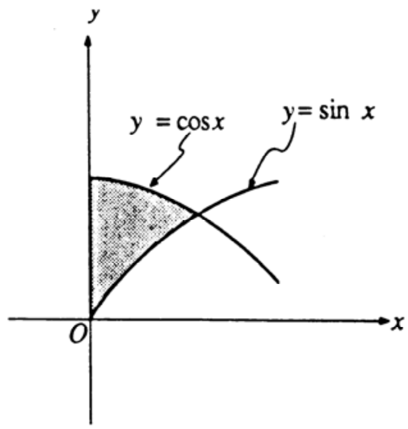
$= (x-2)e^{1-x}$

Point of inflection at $x = 2$.

(d)



1991 BC3



Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of $y = \sin x$ and $y = \cos x$, as shown in the figure above.

(a) Find the area of R .

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) Find the volume of the solid whose base is R and whose cross sections cut by planes perpendicular to the x-axis are squares.

1991 BC3
Solution

(a) $\text{Area} = \int_0^{\pi/4} \cos x - \sin x \, dx$

$= \left(\sin x + \cos x\right)\Big|_0^{\pi/4}$

$= \left(\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\right) - (0+1)$

$= \sqrt{2} - 1$

(b) $V = \pi \int_0^{\pi/4} \cos^2 x - \sin^2 x \, dx$

$= \pi \int_0^{\pi/4} \cos 2x \, dx$

$= \dfrac{\pi}{2} \sin 2x \Big|_0^{\pi/4}$

$= \dfrac{\pi}{2}(1-0) = \dfrac{\pi}{2}$

(c) $V = \int_0^{\pi/4} \left(\cos x - \sin x\right)^2 dx$

$= \int_0^{\pi/4} 1 - 2\sin x \cos x \, dx$

$= \left(x - \sin^2 x\right)\Big|_0^{\pi/4}$

$= \dfrac{\pi}{4} - \dfrac{1}{2} - (0-0)$

$= \dfrac{\pi}{4} - \dfrac{1}{2}$

1991 BC4

Let $F(x) = \int_1^{2x} \sqrt{t^2 + t}\, dt$.

(a) Find $F'(x)$.

(b) Find the domain of F.

(c) Find $\lim_{x \to \frac{1}{2}} F(x)$.

(d) Find the length of the curve $y = F(x)$ for $1 \le x \le 2$.

1991 BC4
Solution

(a) $F'(x) = 2\sqrt{4x^2 + 2x}$

(b) $t^2 + t \geq 0$; therefore $(t \geq 0)$ or $(t \leq -1)$

Since $1 \geq 0$, want $2x \geq 0$

Therefore $x \geq 0$

(c) $\lim_{x \to \frac{1}{2}} F(x) = F\left(\frac{1}{2}\right) = 0$

(d) $L = \int_1^2 \sqrt{1 + (F'(x))^2}\, dx$

$= \int_1^2 \sqrt{1 + 16x^2 + 8x}\, dx$

$= \int_1^2 4x + 1\, dx$

$= 2x^2 + x\Big|_1^2 = 7$

1991 BC5

Let f be the function given by $f(t)=\dfrac{4}{1+t^2}$ and G be the function given by $G(x)=\int_0^x f(t)\,dt$.

(a) Find the first four nonzero terms and the general term for the power series expansion of $f(t)$ about $t=0$.

(b) Find the first four nonzero terms and the general term for the power series expansion of $G(x)$ about $x=0$.

(c) Find the interval of convergence of the power series in part (b). (Your solution must include an analysis that justifies your answer.)

1991 BC5
Solution

(a) $f(t)=\dfrac{4}{1+t^2}$, geometric with $a=4$, $r=-t^2$

$f(t)=4-4t^2+4t^4-4t^6+\cdots+(-1)^n 4t^{2n}+\cdots$

(b) $G(x)=\displaystyle\int_0^x \frac{4}{1+t^2}\,dt=\int_0^x\left(4-4t^2+4t^4-4t^6+\cdots\right)dt$

$$=\left(4t-\frac{4}{3}t^3+\frac{4}{5}t^5-\frac{4}{7}t^7+\cdots+\frac{(-1)^n 4t^{2n+1}}{2n+1}+\cdots\right)\Bigg]_0^x$$

$$=4x-\frac{4}{3}x^3+\frac{4}{5}x^5-\frac{4}{7}x^7+\cdots+\frac{(-1)^n 4x^{2n+1}}{2n+1}+\cdots$$

(c) By Ratio Test,

$$\left|\frac{(-1)^{n+1}4x^{2n+3}}{2n+3}\cdot\frac{2n+1}{(-1)^n 4x^{2n+1}}\right|=\frac{2n+1}{2n+3}x^2$$

$\displaystyle\lim_{n\to\infty}\frac{2n+1}{2n+3}x^2=x^2$; $x^2<1$ for $-1<x<1$

Check endpoints: $G(1)=4-\dfrac{4}{3}+\dfrac{4}{5}-\dfrac{4}{7}+\cdots$ Converges by Alternating Series Test

$G(-1)=-4+\dfrac{4}{3}-\dfrac{4}{5}+\cdots$ Converges by Alternating Series Test

Converges for $-1\le x\le 1$

1991 BC6

A certain rumor spreads through a community at the rate $\frac{dy}{dt} = 2y(1-y)$, where y is the proportion of the population that has heard the rumor at time t.

(a) What proportion of the population has heard the rumor when it is spreading the fastest?

(b) If at time $t = 0$ ten percent of the people have heard the rumor, find y as a function of t.

(c) At what time t is the rumor spreading the fastest?

1991 BC6
Solution

(a) $2y(1-y) = 2y - 2y^2$ is largest when $2 - 4y = 0$

so proportion is $y = \dfrac{1}{2}$

(b) $\dfrac{1}{y(1-y)}dy = 2\,dt$

$\displaystyle\int \frac{1}{y(1-y)}dy = \int 2\,dt$

$\displaystyle\int \frac{1}{y} + \frac{1}{1-y}dy = \int 2\,dt$

$\ln y - \ln(1-y) = 2t + C$

$\ln\dfrac{y}{1-y} = 2t + C$

$\dfrac{y}{1-y} = ke^{2t}$

$y(0) = 0.1 \Rightarrow k = \dfrac{1}{9}$

$y = \dfrac{e^{2t}}{9+e^{2t}}$

(c) $\dfrac{\frac{1}{2}}{1-\frac{1}{2}} = \dfrac{1}{9}e^{2t}$

$1 = \dfrac{1}{9}e^{2t}$

$t = \dfrac{1}{2}\ln 9 = \ln 3$

1992 AB4/BC1

Consider the curve defined by the equation $y + \cos y = x + 1$ for $0 \le y \le 2\pi$.

(a) Find $\frac{dy}{dx}$ in terms of y.

(b) Write an equation for each vertical tangent to the curve.

(c) Find $\frac{d^2y}{dx^2}$ in terms of y.

1992 AB4/BC1
Solution

(a) $\dfrac{dy}{dx} - \sin y \dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx}(1 - \sin y) = 1$

$\dfrac{dy}{dx} = \dfrac{1}{1 - \sin y}$

(b) $\dfrac{dy}{dx}$ undefined when $\sin y = 1$

$y = \dfrac{\pi}{2}$

$\dfrac{\pi}{2} + 0 = x + 1$

$x = \dfrac{\pi}{2} - 1$

(c)
$$\frac{d^2 y}{dx^2} = \frac{d\left(\dfrac{1}{1 - \sin y}\right)}{dx}$$
$$= \frac{-\left(-\cos y \dfrac{dy}{dx}\right)}{(1 - \sin y)^2}$$
$$= \frac{\cos y\left(\dfrac{1}{1 - \sin y}\right)}{(1 - \sin y)^2}$$
$$= \frac{\cos y}{(1 - \sin y)^3}$$

1992 AB5/BC2

Let f be the function given by $f(x) = e^{-x}$, and let g be the function given by $g(x) = kx$, where k is the nonzero constant such that the graph of f is tangent to the graph of g.

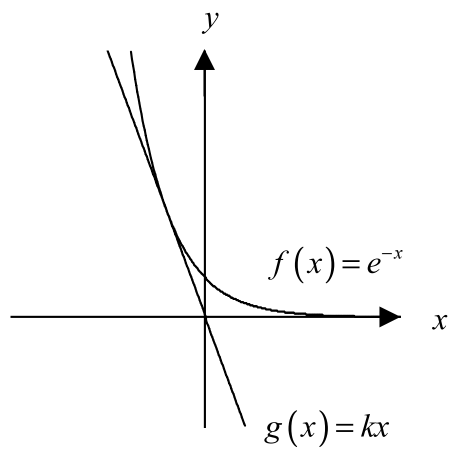
(a) Find the x-coordinate of the point of tangency and the value of k.

(b) Let R be the region enclosed by the y-axis and the graphs of f and g. Using the results found in part (a), determine the area of R.

(c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated by revolving the region R, given in part (b), about the x-axis.

1992 AB5/BC2
Solution

(a)



$f'(x)=-e^{-x};\ g'(x)=k$

$-e^{-x}=k$

$e^{-x}=kx$

$x=-1$ and $k=-e$

(b) $\int_{-1}^{0}\left(e^{-x}-(-ex)\right)dx=\int_{-1}^{0}\left(e^{-x}+ex\right)dx$

$$=\left(-e^{-x}+\frac{ex^2}{2}\right)\Bigg|_{-1}^{0}$$

$$=(-1+0)-\left(-e+\frac{e}{2}\right)$$

$$=\frac{e}{2}-1$$

(c) $\pi\int_{-1}^{0}\left(\left(e^{-x}\right)^2-(-ex)^2\right)dx$

1992 BC3

At time t, $0 \le t \le 2\pi$, the position of a particle moving along a path in the *xy*-plane is given by the parametric equations $x = e^t \sin t$ and $y = e^t \cos t$.

(a) Find the slope of the path of the particle at time $t = \frac{\pi}{2}$.

(b) Find the speed of the particle when $t = 1$.

(c) Find the distance traveled by the particle along the path from $t = 0$ to $t = 1$.

1992 BC3
Solution

(a) $\dfrac{dx}{dt} = e^t \sin t + e^t \cos t$

$\dfrac{dy}{dt} = e^t \cos t - e^t \sin t$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{e^t(\cos t - \sin t)}{e^t(\sin t + \cos t)}$

at $t = \dfrac{\pi}{2}$, $\dfrac{dy}{dx} = \dfrac{e^{\pi/2}(0-1)}{e^{\pi/2}(1+0)} = -1$

(b) speed $= \sqrt{\left(e^t \sin t + e^t \cos t\right)^2 + \left(e^t \cos t - e^t \sin t\right)^2}$

when $t = 1$ speed is

$\sqrt{(e\sin 1 + e\cos 1)^2 + (e\cos 1 - e\sin 1)^2} = e\sqrt{2}$

(c) distance is

$\int_0^1 \sqrt{\left(e^t \sin t + e^t \cos t\right)^2 + \left(e^t \cos t - e^t \sin t\right)^2}\, dt$

$= \int_0^1 \sqrt{2e^{2t}\left(\sin^2 t + \cos^2 t\right)}\, dt = \int_0^1 \sqrt{2}e^t\, dt$

$= \sqrt{2}\, e^t \Big|_0^1 = \sqrt{2}\,(e-1)$

1992 BC4

Let f be a function defined by $f(x) = \begin{cases} 2x - x^2 & \text{for } x \le 1, \\ x^2 + kx + p & \text{for } x > 1. \end{cases}$

(a) For what values of k and p will f be continuous and differentiable at $x = 1$?

(b) For the values of k and p found in part (a), on what interval or intervals is f increasing?

(c) Using the values of k and p found in part (a), find all points of inflection of the graph of f. Support your conclusion.

1992 BC4
Solution

(a) For continuity at $x = 1$,

$\lim_{x\to1^-}\left(2x-x^2\right)=f(1)=\lim_{x\to1^+}\left(x^2+kx+p\right)$

Therefore $1=1+k+p$

Since f is continuous at $x = 1$ and is piecewise polynomial, left and right derivatives exist.

$f_-'(1)=0$ and $f_+'(1)=2+k$

For differentiability at $x = 1$, $0=2+k$.

Therefore $k=-2, \quad p=2$

(b) $f'(x)=2-2x, \quad x\le1$

$2-2x>0$

$x<1$

$f'(x)=2x-2, \ x>1$

$2x-2>0$

$x>1$

Since f increases on each of $(-\infty,1)$ and $(1,\infty)$ and is continuous at $x = 1$, f is increasing on $(-\infty,\infty)$.

(c) $f''(x)=-2, \quad x<1$

$f''(x)=2, \quad x>1$

Since $f''(x)<0$ on $(-\infty,1)$ and

$f''(x)>0$ on $(1,\infty)$ and

$f(1)$ is defined,

$(1, f(1))=(1,1)$ is a point of inflection.

1992 BC5

The length of a solid cylindrical cord of elastic material is 32 inches. A circular cross section of the cord has radius $\frac{1}{2}$ inch.

(a) What is the volume, in cubic inches, of the cord?

(b) The cord is stretched lengthwise at a constant rate of 18 inches per minute. Assuming that the cord maintains a cylindrical shape and a constant volume, at what rate is the radius of the cord changing one minute after the stretching begins? Indicate units of measure.

(c) A force of $2x$ pounds is required to stretch the cord x inches beyond its natural length of 32 inches. How much work is done during the first minute of stretching described in part (b)? Indicate units of measure.

1992 BC5
Solution

(a) $V = \pi r^2 h = \pi\left(\frac{1}{2}\right)^2 \cdot 32 = 8\pi$

(b) $0 = \frac{dV}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt};$

at $t = 1,\ h = 50$ and

so $8\pi = \pi r^2 \cdot 50,$

so $r = \frac{2}{5}$

Therefore $0 = 2\pi\left(\frac{2}{5}\right)(50)\frac{dr}{dt} + \pi\left(\frac{2}{5}\right)^2(18)$

$= \pi\left(40\frac{dr}{dt} + \frac{72}{25}\right)$

$\frac{dr}{dt} = -\frac{9}{125}$ in/min

or

$V = 8\pi = \pi r^2 h,$ so $r = \sqrt{\frac{8}{h}}$

Therefore $\frac{dr}{dt} = \frac{1}{2}\left(\frac{8}{h}\right)^{-\frac{1}{2}} \cdot \left(\frac{-8}{h^2}\right)\cdot\left(\frac{dh}{dt}\right)$

at $t = 1,\ h = 50$ so

$\frac{dr}{dt} = \frac{1}{2}\left(\frac{8}{50}\right)^{-\frac{1}{2}}\left(\frac{-8}{2500}\right)\cdot(18)$

$= -\frac{9}{125}$ in/min

(c) Work $= \int_0^{18} 2x\,dx = x^2\Big|_0^{18} = 18^2$

$= 324$ in-pounds

$= 27$ foot-pounds

1992 BC6

Consider the series $\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)}$, where $p \geq 0$.

(a) Show that the series converges for $p > 1$.

(b) Determine whether the series converges or diverges for $p = 1$. Show your analysis.

(c) Show that the series diverges for $0 \leq p < 1$.

1992 BC6
Solution

(a) $0 < \dfrac{1}{n^p \ln(n)} < \dfrac{1}{n^p}$ for $\ln(n) > 1$, for $n \geq 3$

by p-series test, $\sum \dfrac{1}{n^p}$ converges if $p > 1$

and by direct comparison, $\sum_{n=2}^{\infty} \dfrac{1}{n^p \ln(n)}$ converges.

(b) Let $f(x) = \dfrac{1}{x \ln x}$, so series is $\sum_{n=2}^{\infty} f(n)$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{b \to \infty} \ln\left|\ln x\right|\Big|_2^b = \lim_{b \to \infty} [\ln(\ln(b)) - \ln(\ln 2)] = \infty$$

Since $f(x)$ monotonically decreases to 0, the integral test shows

$\sum_{n=2}^{\infty} \dfrac{1}{n \ln n}$ diverges.

(c) $\dfrac{1}{n^p \ln n} > \dfrac{1}{n \ln n} > 0$ for $p < 1$,

so by direct comparison, $\sum_{n=2}^{\infty} \dfrac{1}{n^p \ln n}$ diverges for $0 \leq p < 1$

1993 AB3/BC1

Consider the curve $y^2 = 4 + x$ and chord AB joining the points $A(-4, 0)$ and $B(0, 2)$ on the curve.

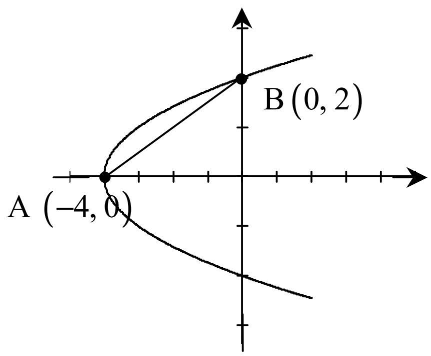
(a) Find the x- and y-coordinates of the point on the curve where the tangent line is parallel to chord AB.

(b) Find the area of the region R enclosed by the curve and the chord AB.

(c) Find the volume of the solid generated when the region R, defined in part (b), is revolved about the x-axis.

1993 AB3/BC1
Solution

(a)



slope of $AB = \frac{2}{4} = \frac{1}{2}$

Method 1: $y = \sqrt{4+x}$; $\frac{dy}{dx} = \frac{1}{2\sqrt{4+x}}$; $\frac{1}{2\sqrt{4+x}} = \frac{1}{2}$

so $x = -3$, $y = 1$

Method 2: $2y\frac{dy}{dx} = 1$; so $2y\left(\frac{1}{2}\right) = 1$ and $y = 1$, $x = -3$

(b)

Method 1: $\int_{-4}^{0}\left(\sqrt{4+x} - \left(\frac{1}{2}x + 2\right)\right)dx = \frac{2}{3}(4+x)^{3/2} - \frac{1}{4}x^2 - 2x\Big|_{-4}^{0}$

$= \frac{2}{3}(4)^{3/2} - (-4+8) = \frac{16}{3} - 4 = \frac{4}{3}$

Method 2: $\int_0^2\left[(2y-4) - (y^2-4)\right]dy = y^2 - \frac{y^3}{3}\Big|_0^2$

$= 4 - \frac{8}{3} = \frac{4}{3}$

Method 3: $\int_{-4}^{0}\sqrt{4+x}\,dx = \frac{16}{3}$; Area of triangle $= 4$

Area of region $= \frac{16}{3} - 4 = \frac{4}{3}$

1993 AB3/BC1
Solution, continued

(c)

<u>Method 1</u>: $\pi\int_{-4}^{0}\left(\left(\sqrt{4+x}\right)^2-\left(\frac{1}{2}x+2\right)^2\right)dx$

$=\pi\int_{-4}^{0}\left(4+x-\left(\frac{1}{4}x^2+2x+4\right)\right)dx$

$=\pi\left(8-\frac{16}{3}\right)=\frac{8\pi}{3}\approx 8.378$

<u>Method 2</u>: $\int_0^2 2\pi y\left[(2y-4)-\left(y^2-4\right)\right]dy=\frac{8\pi}{3}$

<u>Method 3</u>: $\pi\int_{-4}^{0}\left(\sqrt{4+x}\right)^2 dx=\pi\left(4x+\frac{x^2}{2}\right)\Big|_{-4}^{0}$

$=0-\pi(-16+8)=8\pi$

Volume of cone $=\frac{1}{3}\pi(2)^2(4)=\frac{16\pi}{3}$

Volume $=8\pi-\frac{16\pi}{3}=\frac{8\pi}{3}$

1993 AB4/BC3

Let f be the function defined by $f(x) = \ln(2 + \sin x)$ for $\pi \leq x \leq 2\pi$.

(a) Find the absolute maximum value and the absolute minimum value of f. Show the analysis that leads to your conclusion.

(b) Find the x-coordinate of each inflection point on the graph of f. Justify your answer.

1993 AB4/BC3
Solution

(a) $f'(x) = \dfrac{1}{2+\sin x}\cos x;$

In $[\pi, 2\pi]$, $\cos x = 0$ when $x = \dfrac{3\pi}{2}$;

x	$f(x)$
π	$\ln(2) = 0.693$
2π	$\ln(2)$
$\dfrac{3\pi}{2}$	$\ln(1) = 0$

absolute maximum value is ln 2

absolute minimum value is 0

(b) $f''(x) = \dfrac{(-\sin x)(2+\sin x) - \cos x \cos x}{(2+\sin x)^2}$

$= \dfrac{-2\sin x - 1}{(2+\sin x)^2};$

$f''(x) = 0$ when $\sin x = -\dfrac{1}{2}$

$x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

	π to $\dfrac{7\pi}{6}$	$\dfrac{7\pi}{6}$ to $\dfrac{11\pi}{6}$	$\dfrac{11\pi}{6}$ to 2π
sign of f''	−	+	−
concavity	down	up	down

$x = \dfrac{7\pi}{6}$ and $\dfrac{11\pi}{6}$ since concavity changes as indicated at these points

1993 BC2

The position of a particle at any time $t \geq 0$ is given by $x(t) = t^2 - 3$ and $y(t) = \frac{2}{3}t^3$.

(a) Find the magnitude of the velocity vector at $t = 5$.

(b) Find the total distance traveled by the particle from $\boldsymbol{t} = 0$ to $\boldsymbol{t} = 5$.

(c) Find $\frac{\boldsymbol{dy}}{\boldsymbol{dx}}$ as a function of x.

1993 BC 2
Solution

(a) $x'(t)=2t \qquad y'(t)=2t^2$

$x'(5)=10 \qquad y'(5)=50$

$$\|v(5)\|=\sqrt{10^2+50^2}=\sqrt{2600}$$
$$=10\sqrt{26}\approx 50.990$$

(b) $\int_0^5 \sqrt{4t^2+4t^4}\,dt$

$$=\int_0^5 2t\sqrt{1+t^2}\,dt$$
$$=\frac{2}{3}\left(1+t^2\right)^{3/2}\Big|_0^5$$
$$=\frac{2}{3}\left(26^{3/2}-1\right)\approx 87.716$$

(c) $\dfrac{dy}{dx}=\dfrac{y'(t)}{x'(t)}=\dfrac{2t^2}{2t}=t$

$x=t^2-3;\ t^2=x+3$

$t=\sqrt{x+3}$

$$\frac{dy}{dx}=\sqrt{x+3}$$

or

$x=t^2-3;\ t=\sqrt{x+3}$

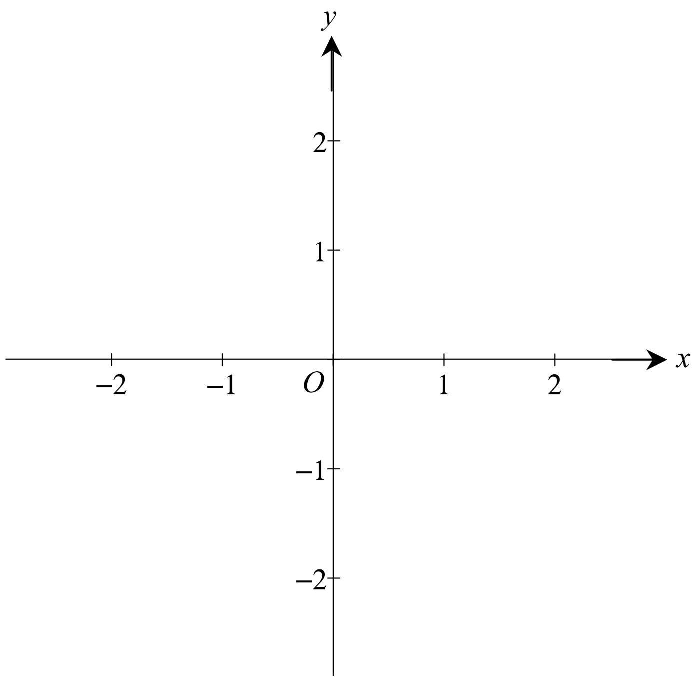
$y=\dfrac{2}{3}t^3;\ y=\dfrac{2}{3}(x+3)^{3/2}$

$$\frac{dy}{dx}=\sqrt{x+3}$$

1993 BC4

Consider the polar curve $\boldsymbol{r} = 2\sin(3\theta)$ for $0 \le \theta \le \pi$.

(a) In the xy-plane provided below, sketch the curve.

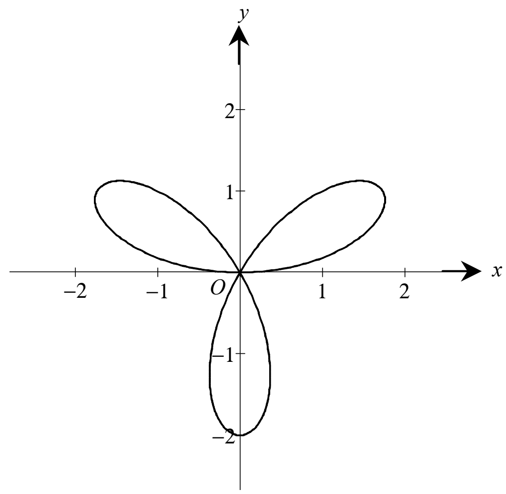


(b) Find the area of the region inside the curve.

(c) Find the slope of the curve at the point where $\theta = \dfrac{\pi}{4}$.

1993 BC4
Solution

(a)



(b) $A = \frac{1}{2}\int_0^{\pi} 4\sin^2 3\theta \, d\theta = \int_0^{\pi} \left(1 - \cos 6\theta\right) d\theta = \left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi} = \pi$

or $\frac{3}{2}\int_0^{\pi/3} 4\sin^2 3\theta \, d\theta = \cdots = \pi$

or $\frac{6}{2}\int_0^{\pi/6} 4\sin^2 3\theta \, d\theta = \cdots = \pi$

(c) $x = 2\sin 3\theta \cos\theta$

$y = 2\sin 3\theta \sin\theta$

$\frac{dx}{d\theta} = -2\sin 3\theta \sin\theta + 6\cos 3\theta \cos\theta$

$\frac{dy}{d\theta} = 2\sin 3\theta \cos\theta + 6\cos 3\theta \sin\theta$

At $\theta = \frac{\pi}{4}$, $\frac{dy}{d\theta} = -2$ and $\frac{dx}{d\theta} = -4$, so

$\frac{dy}{dx} = \frac{-2}{-4} = \frac{1}{2}$

or

$\left(x^2 + y^2\right)^2 = 6x^2 y - 2y^3$

$2\left(x^2 + y^2\right)\left(2x + 2y\frac{dy}{dx}\right) = 6x^2\frac{dy}{dx} + 12xy - 6y^2\frac{dy}{dx}$

At $\theta = \frac{\pi}{4}$, $x = 1$ and $y = 1$ so

$4\left(2 + 2\frac{dy}{dx}\right) = 6\frac{dy}{dx} + 12 - 6\frac{dy}{dx}$

$8 + 8\frac{dy}{dx} = 12 \Rightarrow \frac{dy}{dx} = \frac{1}{2}$

1993 BC5

Let f be the function given by $f(x) = e^{\frac{x}{2}}$.

(a) Write the first four nonzero terms and the general term for the Taylor series expansion of $f(x)$ about $x = 0$.

(b) Use the result from part (a) to write the first three nonzero terms and the general term of the series expansion about $x = 0$ for $g(x) = \dfrac{e^{\frac{x}{2}} - 1}{x}$.

(c) For the function g in part (b), find $g'(2)$ and use it to show that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}$.

1993 BC 5
Solution

(a) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!} + \cdots$

$$e^{x/2} = 1 + \frac{x}{2} + \frac{(x/2)^2}{2!} + \frac{(x/2)^3}{3!} + \cdots + \frac{(x/2)^n}{n!} + \cdots$$

$$= 1 + \frac{x}{2} + \frac{x^2}{2^2 2!} + \frac{x^3}{2^3 3!} + \cdots + \frac{x^n}{2^n n!} + \cdots$$

(b) $$\frac{e^{x/2} - 1}{x} = \frac{\dfrac{x}{2} + \dfrac{x^2}{2^2 2!} + \dfrac{x^3}{2^3 3!} + \cdots + \dfrac{x^n}{2^n n!} + \cdots}{x}$$

$$= \frac{1}{2} + \frac{x}{2^2 2!} + \frac{x^2}{2^3 3!} + \cdots + \frac{x^{n-1}}{2^n n!} + \cdots$$

(c) $$g'(x) = \frac{1}{2^2 2!} + \frac{2x}{2^3 3!} + \cdots + \frac{(n-1)x^{n-2}}{2^n n!} + \cdots$$

$$= \frac{1}{8} + \frac{x}{24} + \cdots + \frac{(n-1)x^{n-2}}{2^n n!} + \cdots$$

$$g'(2) = \frac{1}{2^2 2!} + \frac{2 \cdot 2}{2^3 3!} + \cdots + \frac{(n-1)2^{n-2}}{2^n n!} + \cdots$$

$$= \frac{1}{8} + \frac{1}{12} + \cdots + \frac{n-1}{4n!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{n}{4(n+1)!}$$

Also, $g(x) = \dfrac{e^{x/2} - 1}{x}$

$$g'(x) = \frac{x \cdot \frac{1}{2} e^{x/2} - (1)\left(e^{x/2} - 1\right)}{x^2}$$

$$g'(2) = \frac{2 \cdot \frac{1}{2} e - (e - 1)}{4} = \frac{1}{4}$$

1993 BC6

Let f be a function that is differentiable throughout its domain and that has the following properties.

(i) $f(x+y)=\dfrac{f(x)+f(y)}{1-f(x)f(y)}$ for all real numbers x, y, and $x+y$ in the domain of f

(ii) $\lim\limits_{h\to 0} f(h)=0$

(iii) $\lim\limits_{h\to 0} \dfrac{f(h)}{h}=1$

(a) Show that $f(0)=0$.

(b) Use the definition of the derivative to show that $f'(x)=1+\left[f(x)\right]^2$. Indicate clearly where properties (i), (ii), and (iii) are used.

(c) Find $f(x)$ by solving the differential equation in part (b).

1993 BC6
Solution

(a) Method 1: f is continuous at 0, so $f(0)=\lim_{x\to 0} f(x)=0$

or

Method 2: $f(0)=f(0+0)=\dfrac{f(0)+f(0)}{1-f(0)f(0)}$

$$f(0)\left(1-[f(0)]^2\right)=2f(0)$$
$$f(0)\left(-1-[f(0)]^2\right)=0$$
$$f(0)=0$$

(b) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\dfrac{f(x)+f(h)}{1-f(x)f(h)}-f(x)}{h} \qquad \text{[By (i)]}$$

$$=\lim_{h\to 0}\left[\frac{f(h)}{h}\cdot\frac{1+[f(x)]^2}{1-f(x)f(h)}\right]$$

$$=1\cdot\frac{1+[f(x)]^2}{1-f(x)\cdot 0} \qquad \text{[By (iii) \& (ii)]}$$

$$=1+[f(x)]^2$$

(c) Method 1: Let $y=f(x)$; $\dfrac{dy}{dx}=1+y^2$

$$\frac{dy}{1+y^2}=dx$$
$$\tan^{-1}y=x+C$$
$$y=\tan(x+C)$$
$$f(0)=0\Rightarrow C=0 \quad [\text{or } C=n\pi, n\in\mathbb{Z}]$$
$$f(x)=\tan x \quad [\text{or } f(x)=\tan(x+n\pi)]$$

1993 BC6
Solution, continued

or

Method 2: Guess that $f(x) = \tan x$

$$1 + \left[f(x)\right]^2 = 1 + \tan^2 x = \sec^2 x = f'(x)$$
$$f(0) = \tan(0) = 0$$

Since the solution to the D.E. is unique $f(x) = \tan x$ is the solution.

1994 AB 2-BC 1

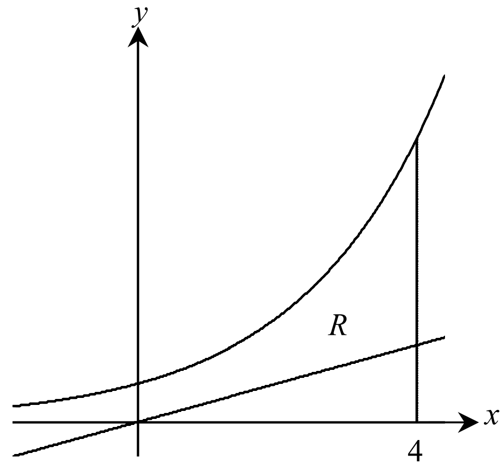
Let ***R*** be the region enclosed by the graphs of $y = e^x$, $y = x$, and the lines ***x*** = 0 and ***x*** = 4.

(a) Find the area of ***R*** .

(b) Find the volume of the solid generated when ***R*** is revolved about the ***x***-axis.

(c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when ***R*** is revolved about the ***y***-axis.

1994 AB 2- BC 1



(a) Area $= \int_0^4 e^x - x\,dx$

$$= e^x - \frac{x^2}{2}\Big|_0^4$$

$$= \left(e^4 - \frac{16}{2}\right) - \left(e^0 - 0\right)$$

$$= e^4 - 9$$

$$\approx 45.598$$

OR <u>Using geometry</u> (area of triangle)

$$\int_0^4 e^x\,dx - \frac{1}{2}\cdot 4 \cdot 4$$

or <u>Using y-axis</u>

$$\int_0^1 y\,dy + \int_1^4 y - \ln y\,dy + \int_4^{e^4} 4 - \ln y\,dy$$

(b) $V_x = \pi \int_0^4 \left(e^x\right)^2 - \left(x\right)^2 dx$

$$= \pi \int_0^4 e^{2x} - x^2\,dx$$

$$= \pi\left(\frac{1}{2}e^{2x} - \frac{x^3}{3}\right)\Big|_0^4$$

$$= \pi\left[\left(\frac{1}{2}e^8 - \frac{64}{3}\right) - \left(\frac{1}{2}e^0 - 0\right)\right]$$

$$= \pi\left(\frac{1}{2}e^8 - \frac{131}{6}\right)$$

$$\approx 1468.646\pi \approx 4613.886$$

1994 AB 2- BC 1 (continued)

[or] Using geometry (Volume of the cone)

$$\pi\int_0^4 \left(e^x\right)^2 dx - \frac{1}{3}\pi \cdot 4^2 \cdot 4$$
$$= \pi\left(\frac{1}{2}e^{2x}\right)\Bigg|_0^4 - \frac{\pi}{3}\cdot 64$$
$$= \pi\left(\frac{1}{2}e^8 - \frac{1}{2}\right) - \frac{64\pi}{3}$$

Using y- axis

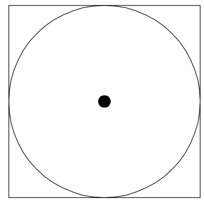
$$2\pi\left[\int_0^1 y\cdot y\,dy + \int_1^4 y\left(y - \ln y\right)dy + \int_4^{e^4} y\left(4 - \ln y\right)dy\right]$$

(c) $V_y = 2\pi\int_0^4 x\left(e^x - x\right)dx$

or

$$V_y = \pi\left[\int_0^1 y^2\,dy + \int_1^4 y^2 - \left(\ln y\right)^2 dy + \int_4^{e^4} 16 - \left(\ln y\right)^2 dy\right]$$

1994 AB 5-BC 2



A circle is inscribed in a square as shown in the figure above. The circumference of the circle is increasing at a constant rate of 6 inches per second. As the circle expands, the square expands to maintain the condition of tangency. (Note: A circle with radius ***r*** has circumference $C = 2\pi r$ and area $A = \pi r^2$)

(a) Find the rate at which the perimeter of the square is increasing. Indicate units of measure.

(b) At the instant when the area of the circle is 25π square inches, find the rate of increase in the area enclosed between the circle and the square. Indicate units of measure.

1994 AB 5-BC 2

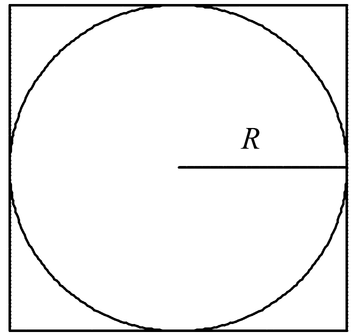
(a) $P = 8R$

$$\frac{dP}{dt} = 8\frac{dR}{dt}$$

$$6 = \frac{dC}{dt} = 2\pi\frac{dR}{dt}$$

$$\frac{dR}{dt} = \frac{3}{\pi};\ \frac{dP}{dt} = \frac{24}{\pi} \text{ inches/second}$$

≈ 7.639 inches/second



(b) $\text{Area} = 4R^2 - \pi R^2$

$$\frac{d(\text{Area})}{dt} = 8R\frac{dR}{dt} - 2\pi R\frac{dR}{dt}$$

$$= (4-\pi)2R\frac{dR}{dt}$$

Area of circle $= 25\pi = \pi R^2$

$$R = 5$$

$$\frac{d(\text{Area})}{dt} = \frac{120}{\pi} - 30 \text{ inches}^2\text{/second}$$

$$= (4-\pi)\frac{30}{\pi} \text{ inches}^2\text{/second}$$

$$\approx 8.197 \text{ inches}^2\text{/second}$$

1994 BC 3

A particle moves along the graph of $\boldsymbol{y} = \cos \boldsymbol{x}$ so that the x-coordinate of acceleration is always 2. At time $\boldsymbol{t} = 0$, the particle is at the point $(\pi, -1)$ and the velocity vector of the particle is $(0, 0)$.

(a) Find the x- and y-coordinates of the position of the particle in terms of t.

(b) Find the speed of the particle when its position is $(4, \cos 4)$.

1994 BC 3

(a) $x''(t) = 2 \Rightarrow x'(t) = 2t + C$

$x'(0) = 0 \Rightarrow C = 0;\ x'(t) = 2t$

$x(t) = t^2 + k,\ x(0) = \pi = k$

$x(t) = t^2 + \pi$

$y(t) = \cos\left(t^2 + \pi\right)$

(b) $\dfrac{dy}{dt} = -2t\sin\left(t^2 + \pi\right)$

$$s(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(2t)^2 + \left(-2t\sin\left(t^2+\pi\right)\right)^2}$$

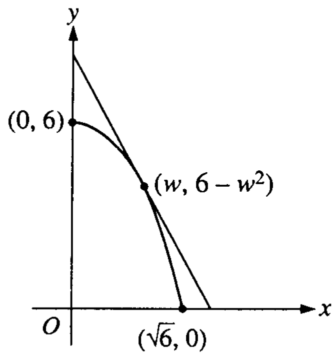
$$= \sqrt{4t^2 + 4t^2\sin^2\left(t^2+\pi\right)}$$

when $x = 4,\ t^2 + \pi = 4;\ t^2 = 4 - \pi$

$$s = \sqrt{4(4-\pi) + 4(4-\pi)\sin^2 4}$$

$$\approx 2.324$$

1994 BC 4



Let $f(x) = 6 - x^2$. For $0 < w < \sqrt{6}$, let $A(w)$ be the area of the triangle formed by the coordinate axes and the line tangent to the graph of f at the point $(w, 6 - w^2)$.

(a) Find $A(1)$.

(b) For what value of w is $A(w)$ a minimum?

1994 BC 4

(a) $f(x) = 6 - x^2$; $f'(x) = -2x$

$f'(1) = -2$

$y - 5 = -2(x-1)$ or $y = -2x + 7$

x int : $\frac{7}{2}$ y int : 7

$A(1) = \frac{1}{2}\left(\frac{7}{2}\right)(7) = \frac{49}{4}$

(b) $f'(w) = -2w$; $y - \left(6 - w^2\right) = -2w(x - w)$

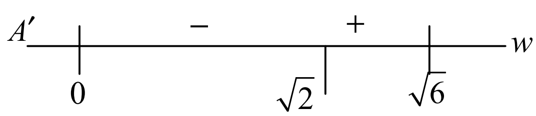
x int: $\frac{6+w^2}{2w}$ y int: $6 + w^2$

$$A(w) = \frac{\left(6+w^2\right)^2}{4w}$$

$$A'(w) = \frac{4w\left(2\left(6+w^2\right)(2w)\right) - 4\left(6+w^2\right)^2}{16w^2}$$

$A'(w) = 0$ when $\left(6+w^2\right)\left(3w^2-6\right) = 0$

$w = \sqrt{2}$



1994 BC 5

Let f be the function given by $f(x) = e^{-2x^2}$.

(a) Find the first four nonzero terms and the general term of the power series for $f(x)$ about $x = 0$.

(b) Find the interval of convergence of the power series for $f(x)$ about $x = 0$. Show the analysis that leads to your conclusion.

(c) Let g be the function given by the sum of the first four nonzero terms of the power series for $f(x)$ about $x = 0$. Show that $\left|f(x) - g(x)\right| < 0.02$ for $-0.6 \le x \le 0.6$.

1994 BC 5

(a) $e^u = 1 + u + \dfrac{u^2}{2} + \dfrac{u^3}{3!} + \cdots + \dfrac{u^n}{n!} + \cdots$

$$e^{-2x^2} = 1 - 2x^2 + \frac{4x^4}{2} - \frac{8x^6}{3!} + \cdots + \frac{(-1)^n 2^n x^{2n}}{n!} + \cdots$$

(b) The series for e^u converges for $-\infty < u < \infty$

So the series for e^{-2x^2} converges for $-\infty < -2x^2 < \infty$

And, thus, for $-\infty < x < \infty$

<u>Or</u>

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1} 2^{n+1} x^{2(n+1)}}{(n+1)!} \cdot \frac{n!}{(-1)^n 2^n x^{2n}}\right|$$

$$= \lim_{n\to\infty} \frac{2}{n+1} x^2 = 0 < 1$$

So the series for e^{-2x^2} converges for $-\infty < x < \infty$

(c) $f(x) - g(x) = \dfrac{16x^8}{4!} - \dfrac{32x^{16}}{5!} + \cdots$

This is an alternative series for each x, since the powers of x are even.

Also, $\left|\dfrac{a_n + 1}{a_n}\right| = \dfrac{2}{n+1} x^2 < 1$ for $-0.6 \le x \le 0.6$ so terms are decreasing in absolute value.

Thus $|f(x) - g(x)| \le \dfrac{16x^8}{4!} \le \dfrac{16(0.6)^8}{4!}$

$= 0.011\cdots < 0.02$

1994 BC 6

Let f and g be functions that are differentiable for all real numbers x and that have the following properties.

(i) $f'(x) = f(x) - g(x)$

(ii) $g'(x) = g(x) - f(x)$

(iii) $f(0) = 5$

(iv) $g(0) = 1$

(a) Prove that $f(x) + g(x) = 6$ for all x.

(b) Find $f(x)$ and $g(x)$. Show your work.

1994 BC 6

(a) $f'(x)+g'(x)=f(x)-g(x)+g(x)-f(x)=0$

so $f+g$ is constant.

$f(0)+g(0)=6,$ so $f(x)+g(x)=6$

(b) $f(x)=6-g(x)$ so

$g'(x)=g(x)-6+g(x)=2g(x)-6$

$\frac{dy}{dx}=2y-6; \ \frac{dy}{2y-6}=dx$

$\frac{1}{2}\ln|2y-6|=x+C$

$\ln|2y-6|=2x+K$

$|2y-6|=e^{2x+K}$

$2y-6=Ae^{2x}$

$x=0 \Rightarrow y=1$ so $-4=A$

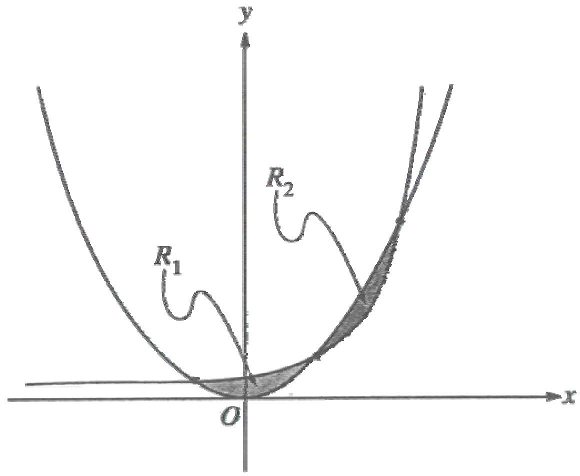
$2y=-4e^{2x}+6$

$y=3-2e^{2x}=g(x)$

$f(x)=6-g(x)=3+2e^{2x}$

Or

1995 AB4/BC2



Note: Figure not drawn to scale.

The shaded regions R_1 and R_2 shown above are enclosed by the graphs of $f(x) = x^2$ and $g(x) = 2^x$.

(a) Find the x- and y-coordinates of the three points of intersection of the graphs of f and g.

(b) Without using absolute value, set up an expression involving one or more integrals that gives the total area enclosed by the graphs of f and g. Do not evaluate.

(c) Without using absolute value, set up an expression involving one or more integrals that gives the volume of the solid generated by revolving the region R_1 about the line $y = 5$. Do not evaluate.

1995 AB4/BC2
Solution

(a) $(2,4)$ $(4,16)$ $(-0.767, 0.588)$ or $(-0.766, 0.588)$

(b) $\int_{-0.767}^{2} \left(2^x - x^2\right) dx + \int_{2}^{4} \left(x^2 - 2^x\right) dx$

or

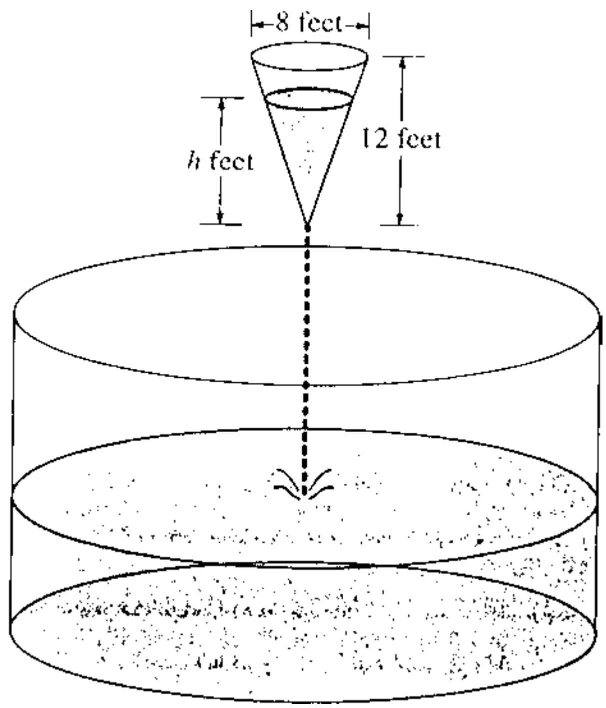
$\int_{0}^{0.588} 2\sqrt{y}\, dy + \int_{0.588}^{4} \left(\sqrt{y} - \frac{\ln y}{\ln 2}\right) dy + \int_{4}^{16} \left(\frac{\ln y}{\ln 2} - \sqrt{y}\right) dy$

(c) $\pi \int_{-0.767}^{2} \left(\left(5 - x^2\right)^2 - \left(5 - 2^x\right)^2\right) dx$

or

$2\pi \int_{0}^{0.588} (5 - y) 2\sqrt{y}\; dy + 2\pi \int_{0.588}^{4} (5 - y)\left(\sqrt{y} - \frac{\ln y}{\ln 2}\right) dy$

1995 AB5/BC3



As shown in the figure above, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h-12)$ feet per minute. (The volume V of a cone with radius r and height h is $V=\frac{1}{3}\pi r^2 h$.)

(a) Write an expression for the volume of water in the conical tank as a function of h.

(b) At what rate is the volume of water in the conical tank changing when $h=3$? Indicate units of measure.

(c) Let y be the depth, in feet, of the water in the cylindrical tank. At what rate is y changing when $h=3$? Indicate units of measure.

1995 AB5/BC3
Solution

(a) $\dfrac{r}{h} = \dfrac{4}{12} = \dfrac{1}{3} \qquad r = \dfrac{1}{3}h$

$V = \dfrac{1}{3}\pi\left(\dfrac{1}{3}h\right)^2 h = \dfrac{\pi h^3}{27}$

(b) $\dfrac{dV}{dt} = \dfrac{\pi h^2}{9}\dfrac{dh}{dt}$

$= \dfrac{\pi h^2}{9}\left(h-12\right) = -9\pi$

V is decreasing at 9π ft^3/min

(c) Let W = volume of water in cylindrical tank

$W = 400\pi y; \quad \dfrac{dW}{dt} = 400\pi\dfrac{dy}{dt}$

$400\pi\dfrac{dy}{dt} = 9\pi$

y is increasing at $\dfrac{9}{400}$ ft/min

1995 BC1

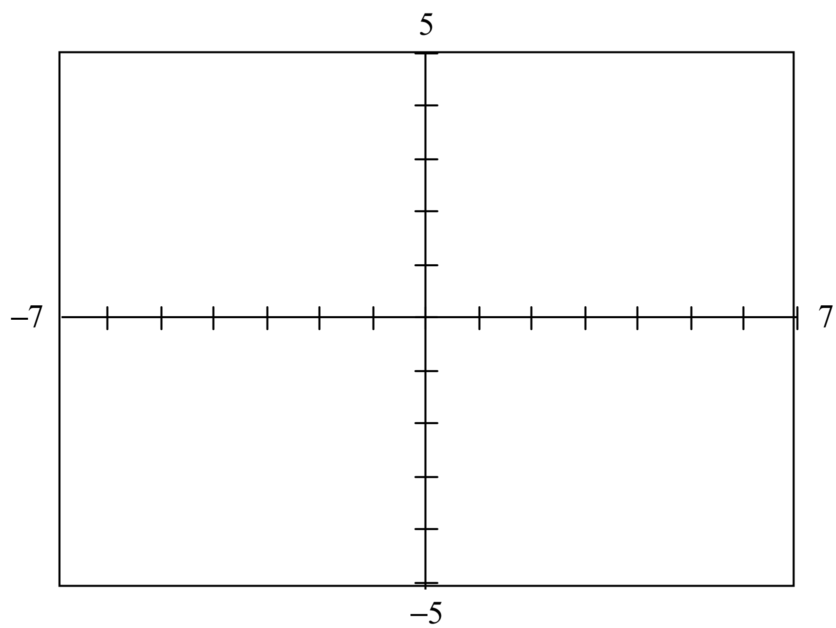
Two particles move in the *xy*-plane. For time $t \geq 0$, the position of particle A is given by $x = t - 2$ and $y = (t-2)^2$, and the position of particle B is given by $x = \frac{3t}{2} - 4$ and $y = \frac{3t}{2} - 2$.

(a) Find the velocity vector for each particle at time $t = 3$.

(b) Set up an integral expression that gives the distance traveled by particle A from $t = 0$ to $t = 3$. Do not evaluate.

(c) Determine the exact time at which the particles collide; that is, when the particles are at the same point at the same time. Justify your answer.

(d) In the viewing window provided below, sketch the paths of particles A and B from $t = 0$ until they collide. Indicate the direction of each particle along its path.



Viewing Window
$[-7, 7] \times [-5, 5]$

1995 BC1
Solution

(a) $V_A = (1, 2t-4)$; $V_A(3) = (1,2)$

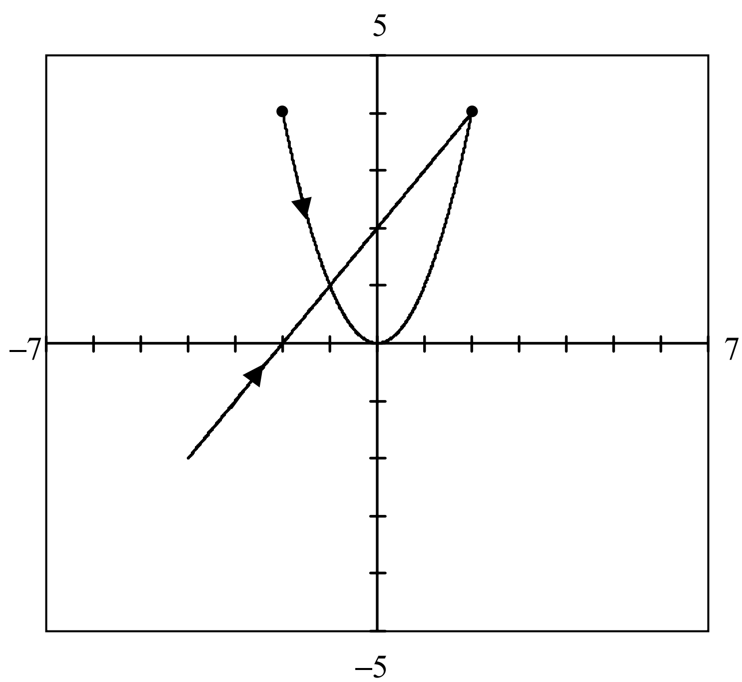
$V_B = \left(\frac{3}{2}, \frac{3}{2}\right)$; $V_B(3) = \left(\frac{3}{2}, \frac{3}{2}\right)$

(b) distance $= \int_0^3 \sqrt{1^2 + (2t-4)^2}\, dt$

(c) Set $t - 2 = \frac{3t}{2} - 4$; $t = 4$

When $t = 4$, the y-coordinates for A and B are also equal. Particles collide at (2,4) when $t = 4$.

(d)



Viewing Window
$[-7, 7] \times [-5, 5]$

1995 BC4

Let f be a function that has derivatives of all orders for all real numbers.
Assume $f(1)=3$, $f'(1)=-2$, $f''(1)=2$, and $f'''(1)=4$.

(a) Write the second-degree Taylor polynomial for f about $x=1$ and use it to approximate $f(0.7)$.

(b) Write the third-degree Taylor polynomial for f about $x=1$ and use it to approximate $f(1.2)$.

(c) Write the second-degree Taylor polynomial for f', the derivative of f, about $x=1$ and use it to approximate $f'(1.2)$.

1995 BC4
Solution

(a) $T_2(x) = 3 + (-2)(x-1) + \frac{2}{2}(x-1)^2$

$f(0.7) \approx 3 + 0.6 + 0.09 = 3.69$

(b) $T_3(x) = 3 - 2(x-1) + (x-1)^2 + \frac{4}{6}(x-1)^3$

$f(1.2) \approx 3 - 0.4 + 0.04 + \frac{2}{3}(0.008) = 2.645$

(c) $T_3'(x) = -2 + 2(x-1) + 2(x-1)^2$

$f'(1.2) \approx -2 + 0.4 + 0.08 = -1.52$

1995 BC5

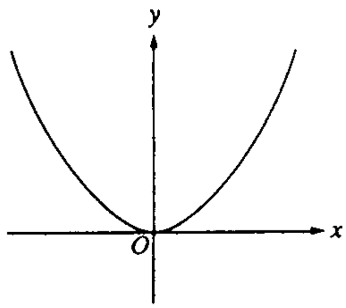


Figure 1
$y = f(x)$

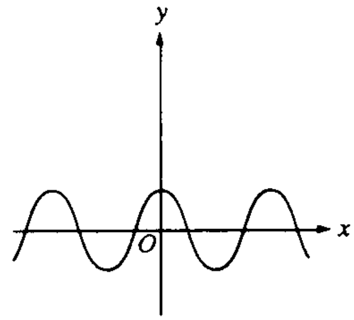


Figure 2
$y = g(x)$

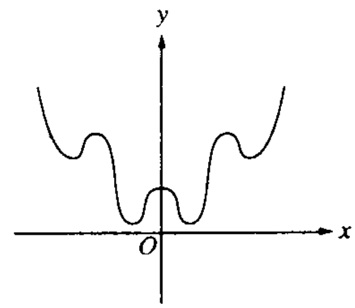
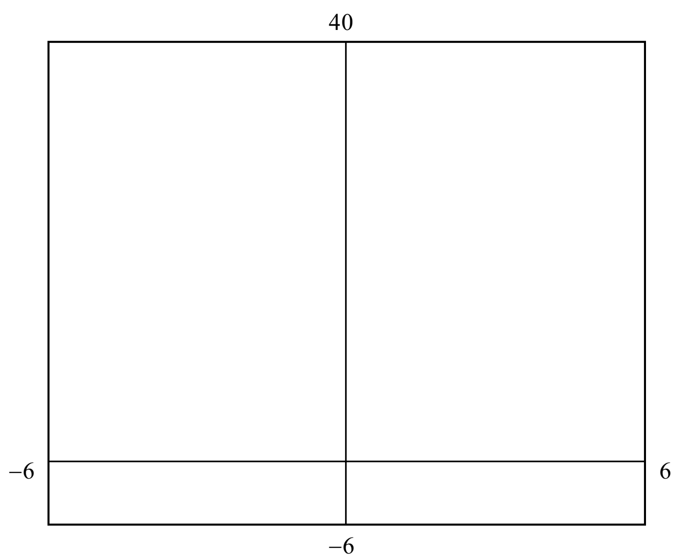


Figure 3

Let $f(x) = x^2$, $g(x) = \cos x$, and $h(x) = x^2 + \cos x$. From the graphs of f and g shown above in Figure 1 and Figure 2, one might think the graph of h should look like the graph in Figure 3.

(a) Sketch the actual graph of h in the viewing window provided below.



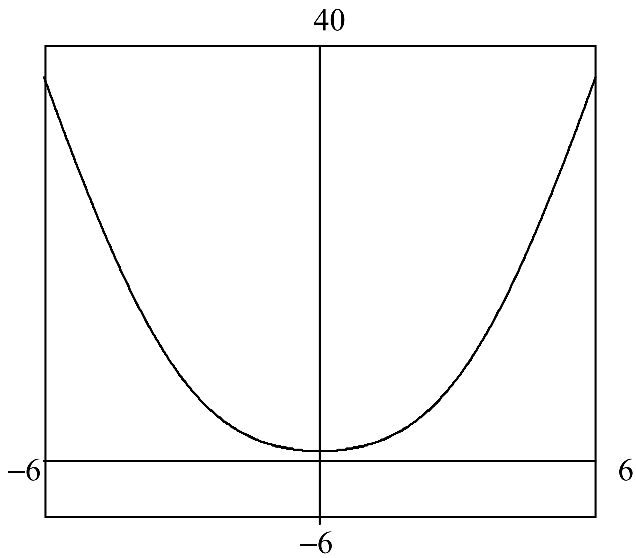
Viewing Window
$[-6, 6]$ x $[-6, 40]$

(b) Use $h''(x)$ to explain why the graph of h does not look like the graph in Figure 3.

(c) Prove that the graph of $y = x^2 + \cos(kx)$ has either no points of inflection or infinitely many points of inflection, depending on the value of the constant k.

1995 BC5
Solution

(a)



(b) $h'(x) = 2x - \sin x$; $h''(x) = 2 - \cos x$

$2 - \cos x > 0$ for all x, so graph must be concave up everywhere

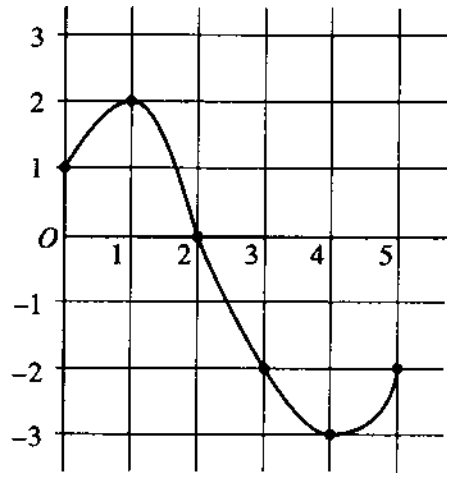
(c) $y''(x) = 2 - k^2 \cos(kx)$

If $k^2 \le 2$, $y'' \ge 0$ for all x, so no inflection points.

If $k^2 > 2$, y'' changes sign and is periodic, so changes sign infinitely many times.

Hence there are infinitely many inflection points.

1995 BC6



Graph of f

Let f be a function whose domain is the closed interval $[0,5]$. The graph of f is shown above.

Let $h(x) = \int_0^{\frac{x}{2}+3} f(t)\,dt$.

(a) Find the domain of h .

(b) Find $h'(2)$.

(c) At what x is $h(x)$ a minimum? Show the analysis that leads to your conclusion.

1995 BC6
Solution

(a) $0 \le \dfrac{x}{2} + 3 \le 5$

$-6 \le x \le 4$

(b) $h'(x) = f\left(\dfrac{x}{2} + 3\right) \cdot \dfrac{1}{2}$

$h'(2) = f(4) \cdot \dfrac{1}{2} = -\dfrac{3}{2}$

(c) h' is positive, then negative, so minimum is at an endpoint

$h(-6) = \int_0^0 f(t)\,dt = 0$

$h(4) = \int_0^5 f(t)\,dt < 0$

since the area below the axis is greater than the area above the axis
therefore minimum at $x = 4$

1996 AB3/BC3

The rate of consumption of cola in the United States is given by $S(t) = Ce^{kt}$, where S is measured in billions of gallons per year and t is measured in years from the beginning of 1980.

(a) The consumption rate doubles every 5 years and the consumption rate at the beginning of 1980 was 6 billion gallons per year. Find C and k.

(b) Find the average rate of consumption of cola over the 10-year time period beginning January 1, 1983. Indicate units of measure.

(c) Use the trapezoidal rule with four equal subdivisions to estimate $\int_5^7 S(t)\,dt$.

(d) Using correct units, explain the meaning of $\int_5^7 S(t)\,dt$ in terms of cola consumption.

1996 AB3/BC3
Solution

(a) $S(t) = Ce^{kt}$

$S(0) = 6 \Rightarrow C = 6$

$S(5) = 12 \Rightarrow 12 = 6e^{5k}$

$2 = e^{5k}$

$k = \dfrac{\ln 2}{5}$ $\quad (0.138 \text{ or } 0.139)$

(b) Average rate $= \dfrac{1}{13-3}\displaystyle\int_3^{13} 6e^{\left(\frac{\ln 2}{5}\right)t}\,dt$

$= \dfrac{3}{\ln 2}\left[e^{2.6\ln 2} - e^{0.6\ln 2}\right]$ billion gal/yr

(19.680 billion gal/yr)

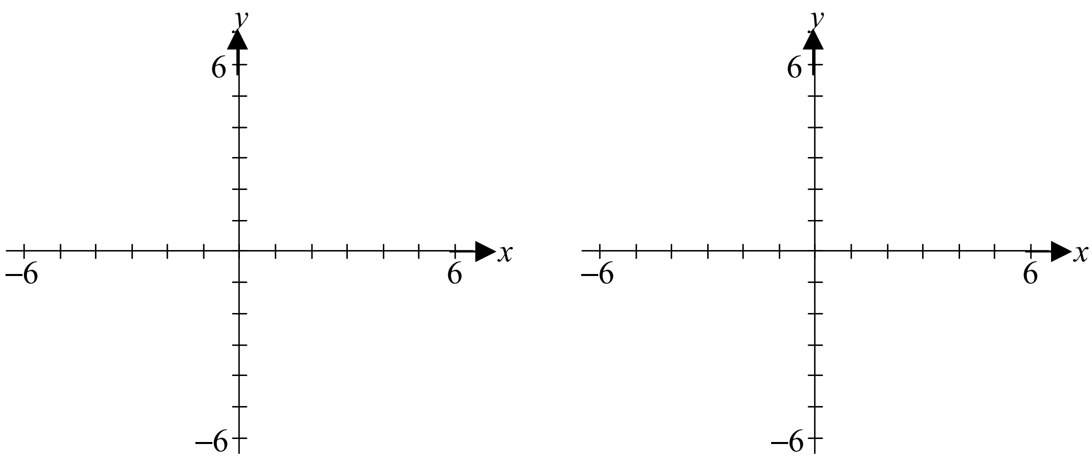
(c) $\displaystyle\int_5^7 S(t)\,dt \doteq \frac{1}{4}\left[S(5) + 2S(5.5) + 2S(6) + 2S(6.5) + S(7)\right]$

(d) This gives the total consumption, in billions of gallons, during the years 1985 and 1986.

1996 AB4/BC4

This problem deals with functions defined by $f(x) = x + b\sin x$, where b is a positive constant and $-2\pi \le x \le 2\pi$.

(a) Sketch the graphs of two of these functions, $y = x + \sin x$ and $y = x + 3\sin x$.



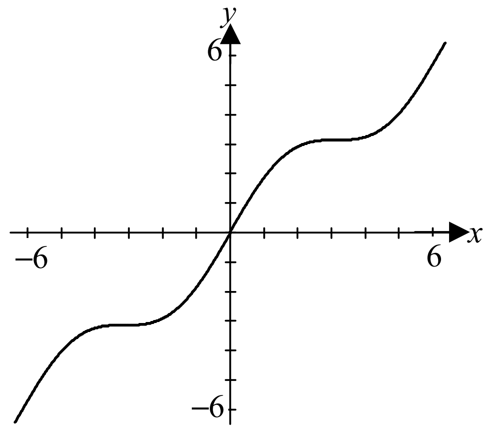
(b) Find the x-coordinates of all points, $-2\pi \le x \le 2\pi$, where the line $y = x + b$ is tangent to the graph of $f(x) = x + b\sin x$.

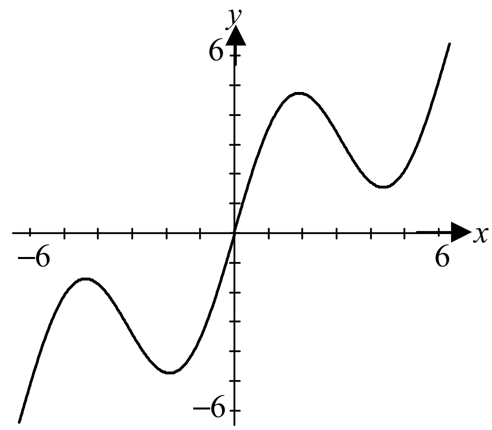
(c) Are the points of tangency described in part (b) relative maximum points of f? Why?

(d) For all values of $b > 0$, show that all inflection points of the graph of f lie on the line $y = x$.

1996 AB4/BC4
Solution

(a)





(b) $y' = 1 = 1 + b\cos x$

$b\cos x = 0$

$\cos x = 0$

$y = x + b = x + b\sin x$

$b = b\sin x$

$1 = \sin x$

$x = -\dfrac{3\pi}{2}$ or $\dfrac{\pi}{2}$

(c) No, because $f'(x) = 1$ $\left(\text{or } f'(x) \neq 0\right)$ at x-coordinates of points of tangency

(d) $f''(x) = -b\sin x = 0$

$\sin x = 0$

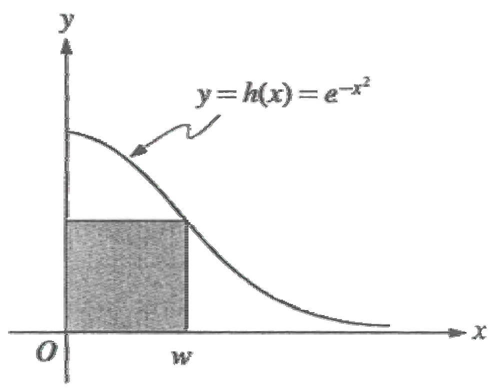
$f(x) = x + b \cdot 0 = x$

at x-coordinates of any inflection points

1996 BC1

Consider the graph of the function h given by $h(x) = e^{-x^2}$ for $0 \le x < \infty$.

(a) Let R be the unbounded region in the first quadrant below the graph of h. Find the volume of the solid generated when R is revolved about the y-axis.



(b) Let $A(w)$ be the area of the shaded rectangle shown in the figure above. Show that $A(w)$ has its maximum value when w is the x-coordinate of the point of inflection of the graph of h.

1996 BC1
Solution

(a) $\text{Volume} = 2\pi \int_0^{\infty} xe^{-x^2}\,dx$

$$= 2\pi \lim_{b\to\infty} \int_0^b xe^{-x^2}\,dx$$

$$= 2\pi \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^b = 2\pi \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2} + \frac{1}{2}e^0\right)$$

$$= 2\pi\left(\frac{1}{2}\right) = \pi$$

or

$$\text{Volume} = \pi \int_0^1 \left(\sqrt{(-\ln y)}\right)^2 dy = -\pi \lim_{a\to 0^+} \int_a^1 (\ln y)\,dy = \pi$$

(b) Maximum:

$A(w) = we^{-w^2}$,

$A'(w) = e^{-w^2} - 2w^2e^{-w^2}$

$\quad = e^{-w^2}\left(1-2w^2\right)$.

$A'(w) > 0$ when $w < \frac{1}{\sqrt{2}}$,

$A'(w) = 0$ when $w = \frac{1}{\sqrt{2}}$,

$A'(w) < 0$ when $w > \frac{1}{\sqrt{2}}$.

Therefore, max occurs when $w = \frac{1}{\sqrt{2}}$

Inflection:

$h(x) = e^{-x^2}, h'(x) = -2xe^{-x^2}$,

$h''(x) = -2e^{-x^2} - 2x(-2x)e^{-x^2}$

$\quad = 2e^{-x^2}\left(-1+2x^2\right)$.

$h''(x) < 0$ when $x < \frac{1}{\sqrt{2}}$,

$h''(x) = 0$ when $x = \frac{1}{\sqrt{2}}$,

$h''(x) > 0$ when $x > \frac{1}{\sqrt{2}}$.

Therefore, inflection point when $x = \frac{1}{\sqrt{2}}$.

Therefore, the maximum value of $A(w)$ and the inflection point of $h(x)$ occur when x and w are $\frac{1}{\sqrt{2}}$.

1996 BC2

The Maclaurin series for $f(x)$ is given by $1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} + \dfrac{x^3}{4!} + \cdots + \dfrac{x^n}{(n+1)!} + \cdots$

(a) Find $f'(0)$ and $f^{(17)}(0)$.

(b) For what values of x does the given series converge? Show your reasoning.

(c) Let $g(x) = x\,f(x)$. Write the Maclaurin series for $g(x)$, showing the first three nonzero terms and the general term.

(d) Write $g(x)$ in terms of a familiar function without using series. Then, write $f(x)$ in terms of the same familiar function.

1996 BC2
Solution

(a) $a_n = \dfrac{f^{(n)}(0)}{n!} = \dfrac{1}{(n+1)!}$

$f'(0) = a_1 = \dfrac{1}{2}$

$f^{(17)}(0) = 17!a_{17} = 17!\left(\dfrac{1}{18!}\right) = \dfrac{1}{18}$

(b)

$$\lim_{n\to\infty} \frac{\left|\dfrac{x^{n+1}}{(n+2)!}\right|}{\left|\dfrac{x^n}{(n+1)!}\right|} = \lim_{n\to\infty} \frac{|x|}{n+2} = 0 < 1$$

Converges for all x , by ratio test

(c) $g(x) = xf(x)$

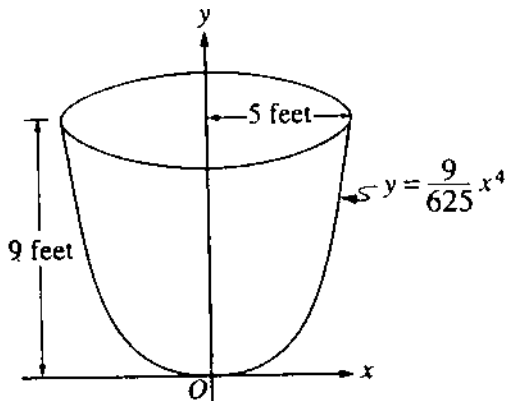
$$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots$$

(d) $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots$

$e^x - 1 = g(x) = xf(x)$

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

1996 BC5



An oil storage tank has the shape as shown above, obtained by revolving the curve $y = \dfrac{9}{625}x^4$ from $x = 0$ to $x = 5$ about the y-axis, where x and y are measured in feet. Oil weighing 50 pounds per cubic foot flowed into an initially empty tank at a constant rate of 8 cubic feet per minute. When the depth of oil reached 6 feet, the flow stopped.

(a) Let h be the depth, in feet, of oil in the tank. How fast was the depth of oil in the tank increasing when $h = 4$? Indicate units of measure.

(b) Find, to the nearest foot-pound, the amount of work required to empty the tank by pumping all of the oil back to the top of the tank.

1996 BC5
Solution

(a) $V = \pi \int_0^h \frac{25}{3}\sqrt{y}\,dy$

$\frac{dV}{dt} = \frac{25\pi}{3}\sqrt{h}\frac{dh}{dt}$

at $h = 4,\ 8 = \frac{25\pi}{3}\sqrt{4}\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{12}{25\pi}$ ft/min

(b) $W = 50\int_0^6 (9-y)\left(\frac{25\pi}{3}\sqrt{y}\right)dy$

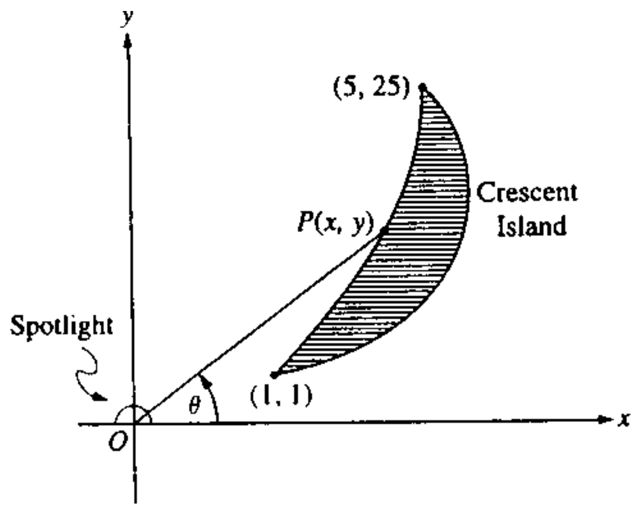
$W = 50\left(\frac{25\pi}{3}\right)\int_0^6 \left(9y^{\frac{1}{2}} - y^{\frac{3}{2}}\right)dy$

$W = 50\left(\frac{25\pi}{3}\right)\left(\frac{2}{3}\cdot 9y^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}}\right)\Bigg|_0^6$

$W = 69{,}257.691$ ft-lbs

to the nearest foot-pound 69,258 ft-lbs

1996 BC6



Note: Figure not drawn to scale.

The figure above shows a spotlight shining on point $P(x, y)$ on the shoreline of Crescent Island. The spotlight is located at the origin and is rotating. The portion of the shoreline on which the spotlight shines is in the shape of the parabola $y = x^2$ from the point $(1,1)$ to the point $(5,25)$. Let θ be the angle between the beam of light and the positive *x*-axis.

(a) For what values of θ between 0 and 2π does the spotlight shine on the shoreline?

(b) Find the *x*- and *y*-coordinates of point P in terms of $\tan\theta$.

(c) If the spotlight is rotating at the rate of one revolution per minute, how fast is the point P traveling along the shoreline at the instant it is at the point $(3,9)$?

1996 BC6
Solution

(a) $\tan\theta_1 = \frac{1}{1} \Rightarrow \theta_1 = \frac{\pi}{4}$ or 0.785

$\tan\theta_2 = \frac{25}{5} \Rightarrow \theta_2 = \tan^{-1} 5$ or 1.373

Therefore, $\frac{\pi}{4} \le \theta \le \tan^{-1} 5$

(b) $\tan\theta = \frac{y}{x} = \frac{x^2}{x} = x$

Therefore, $x = \tan\theta$

$y = x^2 = \tan^2\theta$

(c) $\frac{d\theta}{dt} = 2\pi$

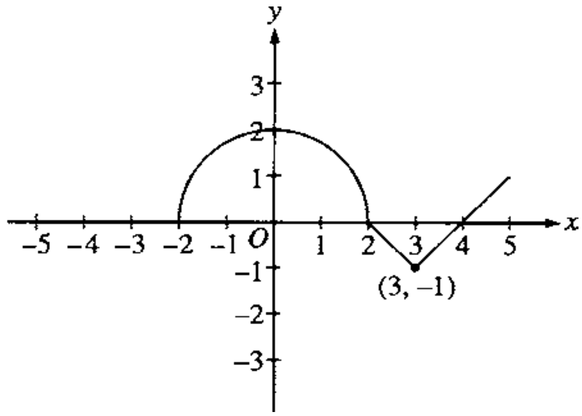
$\frac{dx}{dt} = \sec^2\theta \, \frac{d\theta}{dt}$; $\quad \frac{dy}{dt} = 2\tan\theta \sec^2\theta \, \frac{d\theta}{dt}$

At $(3,9)$: $\frac{dx}{dt} = 10 \cdot 2\pi = 20\pi$

$\frac{dy}{dt} = 2 \cdot 3 \cdot 10 \cdot 2\pi = 120\pi$

$$\text{Speed} = \sqrt{(20\pi)^2 + (120\pi)^2}$$
$$= 20\pi\sqrt{37} \text{ or } 382.191$$

1997 AB5/BC5



The graph of the function f consists of a semicircle and two line segments as shown above. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

(a) Find $g(3)$.

(b) Find all the values of x on the open interval $(-2, 5)$ at which g has a relative maximum. Justify your answer.

(c) Write an equation for the line tangent to the graph of g at $x = 3$.

(d) Find the x-coordinate of each point of inflection of the graph of g on the open interval $(-2, 5)$. Justify your answer.

1997 AB5/BC5
Solution

(a) $g(3) = \int_0^3 f(t)\,dt$

$$= \frac{1}{4} \cdot \pi \cdot 2^2 - \frac{1}{2} = \pi - \frac{1}{2}$$

(b) $g(x)$ has a relative maximum at $x = 2$
because $g'(x) = f(x)$ changes from the positive
to negative at $x = 2$

(c) $g(3) = \pi - \frac{1}{2}$

$g'(3) = f(3) = -1$

$y - \left(\pi - \frac{1}{2}\right) = -1(x - 3)$

(d) graph of g has points of inflection with x-coordinates $x = 0$ and $x = 3$

because $g''(x) = f'(x)$ changes from the positive
to negative at $x = 0$ and from negative to positive at $x = 3$
or
because $g'(x) = f(x)$ changes from increasing
to decreasing at $x = 0$ and from decreasing
to increasing at $x = 3$

1997 AB6/BC6

Let $v(t)$ be the velocity, in feet per second, of a skydiver at time t seconds, $t \geq 0$. After her parachute opens, her velocity satisfies the differential equation $\frac{dv}{dt} = -2v - 32$, with initial condition $v(0) = -50$.

(a) Use separation of variables to find an expression for v in terms of t, where t is measured in seconds.

(b) Terminal velocity is defined as $\lim_{t \to \infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.

(c) It is safe to land when her speed is 20 feet per second. At what time t does she reach this speed?

1997 AB6/BC6
Solution

(a) $\dfrac{dv}{dt} = -2v - 32 = -2(v+16)$

$\dfrac{dv}{v+16} = -2dt$

$\ln|v+16| = -2t + A$

$|v+16| = e^{-2t+A} = e^{A}e^{-2t}$

$v + 16 = Ce^{-2t}$

$-50 + 16 = Ce^{0};\ C = -34$

$v = -34e^{-2t} - 16$

(b) $\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\left(-34e^{-2t} - 16\right) = -16$

(c) $v(t) = -34e^{-2t} - 16 = -20$

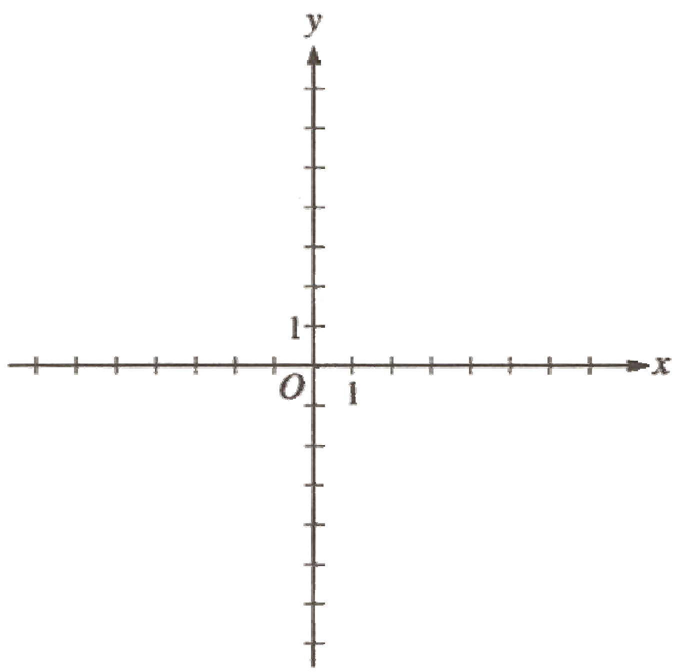
$e^{-2t} = \dfrac{2}{17};\ t = -\dfrac{1}{2}\ln\left(\dfrac{2}{17}\right) = 1.070$

1997 BC1

During the time period from $t = 0$ to $t = 6$ seconds, a particle moves along the path given by $x(t) = 3\cos(\pi t)$ and $y(t) = 5\sin(\pi t)$.

(a) Find the position of the particle when $t = 2.5$.

(b) On the axes provided below, sketch the graph of the path of the particle from $t = 0$ to $t = 6$. Indicate the direction of the particle along its path.



(c) How many times does the particle pass through the point found in part (a)?

(d) Find the velocity vector for the particle at any time t.

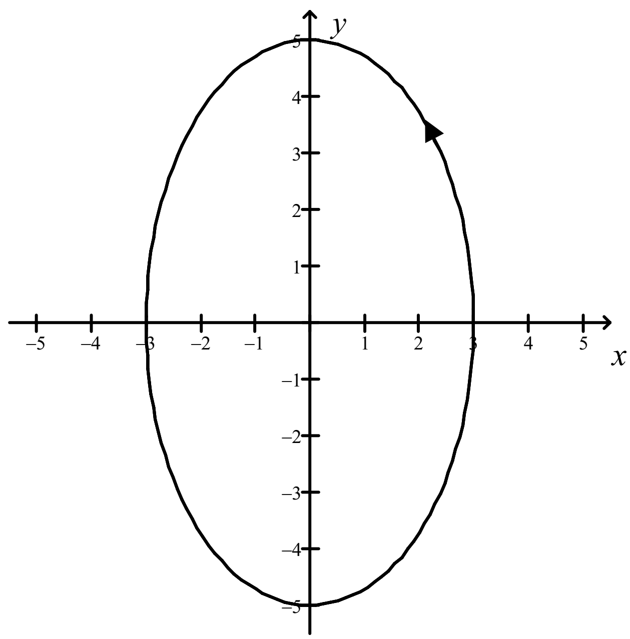
(e) Write and evaluate an integral expression, in terms of sine and cosine, that gives the distance the particle travels from $t = 1.25$ to $t = 1.75$.

1997 BC1
Solution

(a) $x(2.5) = 3\cos(2.5\pi) = 0$

$y(2.5) = 5\sin(2.5\pi) = 5$

(b)



(c) 3

(d) $x'(t) = -3\pi\sin(\pi t) \quad y'(t) = 5\pi\cos(\pi t)$

$\vec{v}(t) = \langle -3\pi\sin(\pi t), 5\pi\cos(\pi t) \rangle$

(e) distance $= \int_{1.25}^{1.75} \sqrt{9\pi^2\sin^2(\pi t) + 25\pi^2\cos^2(\pi t)}\, dt$

$= 5.392$

1997 BC2

Let $P(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$ be the fourth-degree Taylor polynomial for the function f about 4. Assume f has derivatives of all orders for all real numbers.

(a) Find $f(4)$ and $f'''(4)$.

(b) Write the second-degree Taylor polynomial for f' about 4 and use it to approximate $f'(4.3)$.

(c) Write the fourth-degree Taylor polynomial for $g(x) = \int_4^x f(t)\,dt$ about 4.

(d) Can $f(3)$ be determined from the information given? Justify your answer.

1997 BC2
Solution

(a) $f(4) = P(4) = 7$

$$\frac{f'''(4)}{3!} = -2, \quad f'''(4) = -12$$

(b) $P_3(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3$

$P_3'(x) = -3 + 10(x-4) - 6(x-4)^2$

$f'(4.3) \approx -3 + 10(0.3) - 6(0.3)^2 = -0.54$

(c) $P_4(g,x) = \int_4^x P_3(t)\,dt$

$= \int_4^x \left[7 - 3(t-4) + 5(t-4)^2 - 2(t-4)^3\right] dt$

$= 7(x-4) - \frac{3}{2}(x-4)^2 + \frac{5}{3}(x-4)^3 - \frac{1}{2}(x-4)^4$

(d) No. The information given provides values for $f(4), f'(4), f''(4), f'''(4)$ and $f^{(4)}(4)$ only.

1997 BC3

Let R be the region enclosed by the graphs of $y = \ln\left(x^2 + 1\right)$ and $y = \cos x$.

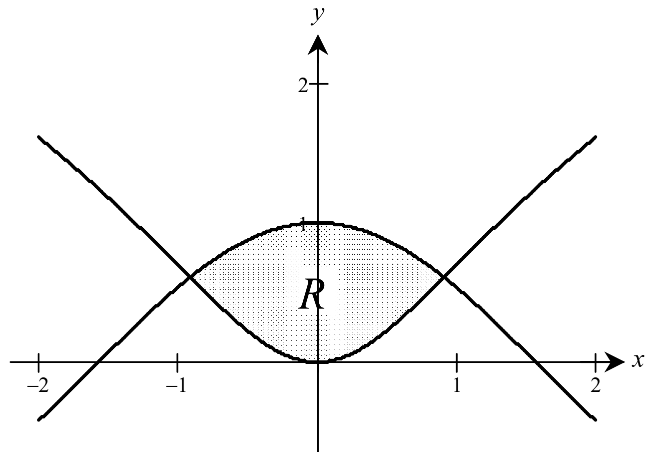
(a) Find the area of R.

(b) Write an expression involving one or more integrals that gives the length of the boundary of the region R. Do not evaluate.

(c) The base of a solid is the region R. Each cross section of the solid perpendicular to the x-axis is an equilateral triangle. Write an expression involving one or more integrals that gives the volume of the solid. Do not evaluate.

1997 BC3
Solution

(a)



$\ln\left(x^2+1\right)=\cos x$

$x=\pm 0.91586$

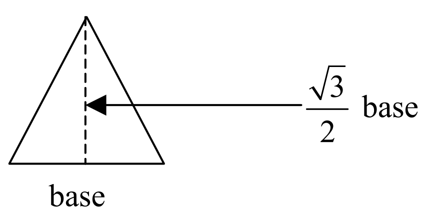
let $B=0.91586$

area $=\int_{-B}^{B}\left[\cos x-\ln\left(x^2+1\right)\right]dx$

$=1.168$

(b) $L=\int_{-B}^{B}\sqrt{1+\left(\dfrac{2x}{x^2+1}\right)^2}\,dx+\int_{-B}^{B}\sqrt{1+\left(-\sin x\right)^2}\,dx$

(c)



area of cross section $=\dfrac{1}{2}\left[\cos x-\ln\left(x^2+1\right)\right]\times\left[\dfrac{\sqrt{3}}{2}\left(\cos x-\ln\left(x^2+1\right)\right)\right]$

volume $=\displaystyle\int_{-B}^{B}\frac{\sqrt{3}}{4}\left[\cos x-\ln\left(x^2+1\right)\right]^2 dx$

1997 BC4

Let $x = ky^2 + 2$, where $k > 0$.

(a) Show that for all $k > 0$, the point $\left(4, \sqrt{\frac{2}{k}}\right)$ is on the graph of $x = ky^2 + 2$.

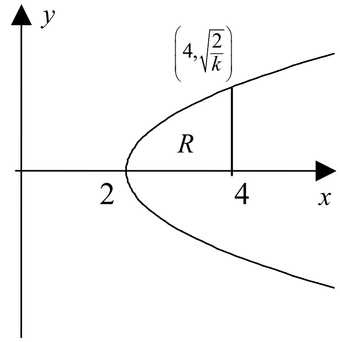
(b) Show that for all $k > 0$, the tangent line to the graph of $x = ky^2 + 2$ at the point $\left(4, \sqrt{\frac{2}{k}}\right)$ passes through the origin.

(c) Let R be the region in the first quadrant bounded by the x-axis, the graph of $x = ky^2 + 2$, and the line $x = 4$. Write an integral expression for the area of the region R and show that this area decreases as k increases.

1997 BC4
Solution

(a) $4 = k\left(\frac{2}{k}\right) + 2$

$4 = 4$



(b) $x = ky^2 + 2$

$1 = 2ky\frac{dy}{dx}$

$\left.\frac{dy}{dx}\right|_{y=\sqrt{2/k}} = \frac{1}{2\sqrt{2k}}$

the tangent line is

$y - \sqrt{\frac{2}{k}} = \frac{1}{2\sqrt{2k}}(x - 4)$

$y = \frac{1}{2\sqrt{2k}}x$ which contains $(0,0)$

or

slope of the line through $(0,0)$ and $\left(4, \sqrt{2/k}\right)$ is $\frac{\sqrt{2/k}}{4} = \frac{1}{2\sqrt{2k}}$

which is the same as the slope of the tangent line

(c) $A = \int_0^{\sqrt{2/k}} \left(4 - \left(ky^2 + 2\right)\right) dy$

or

$A = \frac{1}{\sqrt{k}} \int_2^4 \sqrt{x-2}\, dx$

$A = \frac{4\sqrt{2}}{3} k^{-0.5}$

$\frac{dA}{dk} = -\frac{2\sqrt{2}}{3} k^{-1.5} < 0$ for all $k > 0$

thus the area decreases as k increases